

Solutions to exercises 2.

1. Calculate the first three cumulants $\langle X^n \rangle_c$ ($n = 1, 2, 3$) for a normally distributed random number X with mean μ and variance σ^2 .

The characteristic function is the Fourier-transform of the probability distribution.

$$\Phi(k) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} dx e^{ikx} e^{-(x-\mu)^2/2\sigma^2}$$

we do the variable transformation $y = (x - \mu)/\sigma$ which gives us

$$\Phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy e^{ik(\sigma y + \mu)} e^{-y^2/2} = e^{ik\mu - k^2\sigma^2/2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy e^{-(ik\sigma - y)^2/2}$$

where the integral is simply $\sqrt{2\pi}$. Now we see that the logarithm of the characteristic function is simply given by

$$\ln \Phi(k) = ik\mu - k^2\sigma^2/2$$

Identifying the powers of k gives $\langle X \rangle_c = \mu$, $\langle X^2 \rangle_c = \sigma^2$ and $\langle X^n \rangle_c = 0$ for $n > 2$.

2. Show how a random number with frequency function

$$p_a(x) = \begin{cases} (1+a)x^a & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \quad (a > -1)$$

can be obtained from a rectangularly distributed random number.

The cumulative distributions are

$$P_X(x) = \int_0^x (1+a)t^a dt \quad \text{and} \quad P_R(r) = r$$

Setting them equal gives

$$r = P_X(x) = x^{a+1} \Rightarrow x = r^{\frac{1}{a+1}}$$

3. Construct a normally distributed random number with mean μ and standard deviation σ from a normally distributed random number with mean 0 and unit variance.

The cumulative distributions are

$$P_X(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt \quad \text{and} \quad P_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^y e^{-(t-\mu)^2/2\sigma^2} dt$$

Setting them equal and making the variable substitution $t = (t - \mu)/\sigma$ gives

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(y-\mu)/\sigma} e^{-t^2/2} dt$$

and we don't have to perform the integrations to see that $y = x\sigma + \mu$.

4. Show how a random number with frequency function

$$p(x) = \begin{cases} 4/[\pi(1+x^2)] & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

can be obtained from rectangularly distributed random numbers by (a) the transformation method, and (b) the accept/reject method.

(a) Using the standard variable substitution $t' = \tan(t)$ in the cumulative distribution we get

$$r = \frac{4}{\pi} \int_0^x \frac{dt}{1+t^2} = \frac{4}{\pi} \tan^{-1}(x)$$

and $x = \tan(r\pi/4)$.

(b) The simplest overestimating function is $g(x) = 4/\pi$. So we generate a flat random number between 0 and 1 and accept it with the probability $\frac{1}{1+x^2}$. The efficiency of this method is obviously larger than 50% ($\pi/4$ to be precise).

5. If R_1 och R_2 are independent and uniformly distributed random numbers between 0 and 1, then $X_1 = \sqrt{-2 \ln R_1} \cos 2\pi R_2$ and $X_2 = \sqrt{-2 \ln R_1} \sin 2\pi R_2$ are independent and normally distributed random numbers with mean 0 and unit variance (the Box-Muller method). Suppose R_1 and R_2 are consecutive numbers obtained by the linear congruential method

$$\begin{aligned} R_j &= I_j/m \\ I_j &= aI_{j-1} + c \pmod{m} \end{aligned}$$

where a , c , m and all I_j are integers. Show that the corresponding points (X_1, X_2) fall onto the curve

$$\begin{cases} x_1 = \rho \cos \phi(\rho) \\ x_2 = \rho \sin \phi(\rho) \end{cases}$$

where $\phi(\rho) = 2\pi[a \exp(-\rho^2/2) + c/m]$, $\rho \geq 0$.

We can write subsequent I_j as

$$I_{j+1} + n \cdot m = aI_j + c$$

and we simply get

$$\rho = \sqrt{-2 \ln I_j/m} \quad \text{and} \quad \phi = 2\pi(aI_j + c - n \cdot m)/m$$

Which gives

$$\phi = 2\pi \left(a e^{-\rho^2/2} + c/m \right) [-n2\pi]$$