

Solutions to exercises 3

1. Importance sampling. One wants to make a Monte Carlo estimate of the integral

$$\int_0^1 \frac{dx}{\sqrt{x}}$$

based on random numbers X with frequency function

$$p_a(x) = \begin{cases} (1-a)x^{-a} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases},$$

where $0 < a < 1$. Calculate the mean and variance of $Y = 1/[p_a(X)\sqrt{X}]$ as functions of a .

$$\langle Y \rangle = \int_0^1 \frac{dx}{p_a(x)\sqrt{x}} p_a(x) = \int_0^1 \frac{dx}{\sqrt{x}} = 2$$

$$\langle Y^2 \rangle = \int_0^1 \frac{dx}{(p_a(x)\sqrt{x})^2} p_a(x) = \int_0^1 \frac{dx}{p_a(x)x} = \frac{1}{a(1-a)}$$

$$\sigma = \langle Y^2 \rangle - \langle Y \rangle^2 = \frac{(1-2a)^2}{a(1-a)}$$

2. Consider a stationary two-state Markov chain $X_0, X_1, \dots$ ($X_n \in \{1, 2\}$) with transition probabilities $W_{ij} = P\{X_{n+1} = i | X_n = j\}$, $i, j \in \{1, 2\}$. Let $\mathbf{p}(n)$ be the probability distribution after step n , that is the vector with components $P\{X_n = 1\}$ and $P\{X_n = 2\}$.

- (a) Show that the transition matrix $\mathbf{W}$ can be written as

$$\mathbf{W} = \begin{pmatrix} \alpha & 1-\beta \\ 1-\alpha & \beta \end{pmatrix}$$

for some $0 \leq \alpha, \beta \leq 1$.

For any given initial state the probability ($\{1, 2\}$), the sum of probabilities of reaching any final state must be one. So, $W_{11} + W_{12} = 1$ and $W_{21} + W_{22} = 1$, and since all elements should be positive we can always parameterize it as above.

- (b) Solve the eigenvalue problem $\mathbf{W}\mathbf{v} = \lambda\mathbf{v}$.

We find the eigen vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1-\beta \\ 1-\alpha \end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

with eigenvalues $\lambda_1 = 1$ and $\lambda_2 = \alpha + \beta - 1$

(c) Show that unless $\alpha = \beta = 0$ or $\alpha = \beta = 1$

$$\mathbf{p}(n) = \mathbf{W}^n \mathbf{p}(0) \rightarrow \frac{1}{2 - \alpha - \beta} \begin{pmatrix} 1 - \beta \\ 1 - \alpha \end{pmatrix} \quad n \rightarrow \infty$$

We can write $\mathbf{p}(0) = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2$. Since this is a probability distribution the sum of the components must be equal to 1 which means $c_1 = 1/(2 - \alpha - \beta)$. We now immediately get

$$\mathbf{p}(n) = c_1 \lambda_1^n \mathbf{v}_1 + c_2 \lambda_2^n \mathbf{v}_2$$

and if $0 < \alpha, \beta < 1$, we have $|\lambda_2| = |\alpha + \beta - 1| < 1$, so in the limit $n \rightarrow \infty$ we have

$$\mathbf{p}(n) \rightarrow c_1 \lambda_1^n \mathbf{v}_1 = c_1 \mathbf{v}_1 = \frac{1}{2 - \alpha - \beta} \begin{pmatrix} 1 - \beta \\ 1 - \alpha \end{pmatrix}$$

(d) How does $\mathbf{p}(n)$ behave in the remaining cases $\alpha = \beta = 0$ and $\alpha = \beta = 1$?

For $\alpha = \beta = 0$, $\mathbf{p}(n)$ will oscillate between two probability distributions. For $\alpha = \beta = 1$, the initial distribution will not change.