

Solutions to Exercises 5

1. Use Euler's method to solve $y'(x) = -y$, $y(0) = 1$.

(a) Determine y after n steps of size h , y_n . What is the exact solution?

$$\begin{aligned}y(x) &= e^{-x} \\y(0) &= 1 \\y(0+h) &= 1-h \\y(0+2h) &= (1-h)^2 \\y(0+nh) &= (1-h)^n\end{aligned}$$

(b) For which values of h is the sequence $y_0, y_1, \dots$ bounded?

$$h < 2$$

(c) Let $y(x, h)$ denote the estimate of $y(x)$ obtained for step size h . Determine the global error $y(x, h) - y(x)$ to leading order in h , for fixed x .

We have $y(x, h) = (1-h)^{x/h}$, and $y(x, h) - y(x) \rightarrow 0$ for $h \rightarrow 0$. If we Taylor expand the difference around $h = 0$ we get

$$\frac{dy(x, h)}{dh} = -(1-h)^{x/h} \left(\frac{x}{h(1-h)} + \frac{x \ln(1-h)}{h^2} \right) \rightarrow -xe^{-x}/2 \text{ as } h \rightarrow 0$$

$$y(x, h) - y(x) \approx h \cdot -xe^{-x}/2 + \mathcal{O}(h^2)$$

2. The midpoint method applied to the equation $y'(x) = -y$ gives the recursion formula $y_{n+1} = y_{n-1} - 2hy_n$, which has a general solution of the form $y_n = A\lambda_1^n + B\lambda_2^n$. Determine λ_1 och λ_2 .

$$\lambda_{1,2} = \pm \sqrt{1+h^2} - h$$

3. Show that the last step in the modified midpoint method,

$$y(x_0 + Nh, h) = \frac{1}{2} (y_N + y_{N-1} + hf(x_N, y_N)) ,$$

can be written as

$$y(x_0 + Nh, h) = \frac{1}{4} (y_{N-1} + 2y_N + y_{N+1}) ,$$

where $y_{N+1} = y_{N-1} + 2hf(x_N, y_N)$ (corresponding to one extra step with the non-modified midpoint method). This shows that an oscillating component (see exercise 2) is damped by this last step.

$hf(x_N, y_N) = \frac{1}{2}(y_{N+1} - y_{N-1})$ so

$$y(x_0 + Nh, h) = \frac{1}{2} (y_N + y_{N-1} + hf(x_N, y_N)) = \frac{1}{4} (y_{N-1} + 2y_N + y_{N+1})$$

For small h , the term $\lambda_2^n \rightarrow (-1)^n$ which just oscillates between -1 and 1 . But the last setp for this term becomes

$$\propto \lambda_2^{N-1} \left(1 + 2\lambda_2 + \lambda_2^2\right) = \lambda_2^{N-1} \left(1 - \sqrt{1+h^2} - h\right)^2 \rightarrow 0 \text{ as } h \rightarrow 0$$