

Solutions to Exercises 6.

1. Consider a one-dimensional harmonic oscillator with energy

$$H(x, p) = \frac{1}{2} (p^2 + x^2)$$

$$(m = 1, V = x^2/2).$$

- (a) Write the leapfrog equations

$$\begin{aligned} x(h/2) &= x(0) + \frac{h}{2}p(0) \\ p(h) &= p(0) - hV'(x(h/2)) \\ x(h) &= x(h/2) + \frac{h}{2}p(h) \end{aligned}$$

in the vector-matrix form

$$\begin{pmatrix} x(h) \\ p(h) \end{pmatrix} = \mathbf{T} \begin{pmatrix} x(0) \\ p(0) \end{pmatrix},$$

and compute the determinant of the 2×2 -matrix $\mathbf{T}$.

With $V'(x) = x$ the matrix becomes

$$\mathbf{T} = \begin{pmatrix} 1 - \frac{h^2}{2} & , & h(1 - \frac{h^2}{4}) \\ -h & , & 1 - \frac{h^2}{2} \end{pmatrix}$$

which gives the determinant 1.

- (b) Same thing for the Euler method.

For the Euler method we have

$$\begin{aligned} x(h) &= x(0) + hp(0) \\ p(h) &= p(0) - hV'(x(0)) = p(0) - hx(0) \end{aligned}$$

and the matrix becomes

$$\mathbf{T} = \begin{pmatrix} 1 & , & h \\ -h & , & 1 \end{pmatrix}$$

giving the determinant $1 + h^2$

- (c) Same thing for the implicit Euler method.

For the implicit Euler method we have

$$\begin{aligned} x(h) &= x(0) + hp(h) \\ p(h) &= p(0) - hx(h) \end{aligned}$$

and the matrix becomes

$$\mathbf{T} = \begin{pmatrix} \frac{1}{1+h^2} & , & \frac{h}{1+h^2} \\ -\frac{h}{1+h^2} & , & \frac{1}{1+h^2} \end{pmatrix}$$

giving the determinant $1/(1+h^2)$

(d) Same thing for the exact solution.

We have the general solution

$$\begin{aligned} x(h) &= x(0) \cos(h) + p(0) \sin(h) \\ p(h) &= p(0) \cos(h) - x(0) \sin(h) \end{aligned}$$

and the matrix becomes

$$\mathbf{T} = \begin{pmatrix} \cos(h) & , & \sin(h) \\ -\sin(h) & , & \cos(h) \end{pmatrix}$$

giving, of course, the determinant 1