

Solutions to Exercises 7.

1. Consider the initial value problem

$$\frac{d\mathbf{u}}{dt} = \mathbf{A}\mathbf{u} \quad \mathbf{u}(0) = \sum_{k=1}^N c_k \mathbf{v}^{(k)} \quad (1)$$

where the $\mathbf{v}^{(k)}$'s are eigenvectors of the (t -independent) matrix $\mathbf{A}$ with eigenvalues λ_k .

- (a) Verify that the exact solution is given by

$$\mathbf{u}(t) = \sum_{k=1}^N c_k e^{\lambda_k t} \mathbf{v}^{(k)}$$

Clearly $\mathbf{u}(0) = \sum_{k=1}^N c_k \mathbf{v}^{(k)}$. Then we have

$$\frac{d\mathbf{u}}{dt} = \sum_{k=1}^N c_k \frac{d}{dt} e^{\lambda_k t} \mathbf{v}^{(k)} = \sum_{k=1}^N c_k \lambda_k e^{\lambda_k t} \mathbf{v}^{(k)}$$

which is equal to

$$\mathbf{A}\mathbf{u} = \sum_{k=1}^N c_k e^{\lambda_k t} \mathbf{A}\mathbf{v}^{(k)} = \sum_{k=1}^N c_k e^{\lambda_k t} \lambda_k \mathbf{v}^{(k)}$$

- (b) Show that the Crank-Nicholson formula $\mathbf{u}^{n+1} - \mathbf{u}^n = h(\mathbf{A}\mathbf{u}^n + \mathbf{A}\mathbf{u}^{n+1})/2$ gives

$$\mathbf{u}^n = \sum_{k=1}^N c_k \left(\frac{1 + h\lambda_k/2}{1 - h\lambda_k/2} \right)^n \mathbf{v}^{(k)}$$

Shuffling around gives

$$\mathbf{u}^{n+1} = (\mathbf{I} - \frac{h}{2}\mathbf{A})^{-1}(\mathbf{I} + \frac{h}{2}\mathbf{A})\mathbf{u}^n = [(\mathbf{I} - \frac{h}{2}\mathbf{A})^{-1}(\mathbf{I} + \frac{h}{2}\mathbf{A})]^{n+1}\mathbf{u}^0$$

and insertion of the expansion of $\mathbf{u}^0$ gives the desired formula.

2. Consider the heat equation $\partial u / \partial t = \partial^2 u / \partial x^2$ with boundary conditions $u(0, t) = u(1, t) = 0$. The approximation $\partial^2 u / \partial x^2 \approx (u_{j+1} - 2u_j + u_{j-1})/a^2$ leads to a matrix equation of the same type as in (1) for $\mathbf{u} = (u_1, \dots, u_{J-1})$. Show that

$$\mathbf{v}^{(k)} = \left(\sin \frac{k\pi}{J}, \sin \frac{2k\pi}{J}, \dots, \sin \frac{(J-1)k\pi}{J} \right) \quad (k = 1, \dots, J-1)$$

are eigenvectors of the corresponding matrix $\mathbf{A}$ with eigenvalues $\lambda_k = -2[1 - \cos(k\pi/J)]/a^2$.

The components of $\mathbf{v}^{(k)}$ are

$$\mathbf{v}_j^{(k)} = \sin \frac{jk\pi}{J},$$

which are also correctly defined for $j = 0$ and $j = J$. Multiplying with $\mathbf{A}$ gives

$$\begin{aligned} [\mathbf{A}\mathbf{v}^{(k)}]_j &= \frac{1}{a^2} \left(\sin \frac{(j-1)k\pi}{J} + \sin \frac{(j+1)k\pi}{J} - 2 \sin \frac{jk\pi}{J} \right) \\ &= \frac{1}{a^2} \left(2 \sin \frac{jk\pi}{J} \cos \frac{k\pi}{J} - 2 \sin \frac{jk\pi}{J} \right) = \frac{2}{a^2} \left(\cos \frac{k\pi}{J} - 1 \right) \mathbf{v}_j^{(k)} \end{aligned}$$

3. Let $u_j^n = u(x_j, t_n)$, $x_j = ja$, $t_n = nh$. Show that the equation

$$\frac{u_j^{n+1} - u_j^{n-1}}{2h} = \frac{u_{j+1}^n - u_j^{n+1} - u_j^{n-1} + u_{j-1}^n}{a^2}$$

approximates the heat equation $\partial u / \partial t = \partial^2 u / \partial x^2$ if $h = a^2$ and $a \rightarrow 0$, but that it approximates the equation $\partial u / \partial t = \partial^2 u / \partial x^2 - \partial^2 u / \partial t^2$ if $h = a$ and $a \rightarrow 0$.

If we add and remove $2u_j^n/a^2$ on the right we get

$$\frac{u_j^{n+1} - u_j^{n-1}}{2h} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{a^2} - \frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{a^2}$$

The left hand side is always $\partial u / \partial t$ as $h \rightarrow 0$ and the first term on the right is always $\partial^2 u / \partial x^2$ as $a \rightarrow 0$. The second term is $-\partial^2 u / \partial t^2$ if $h = a \rightarrow 0$, but is zero if $h = a^2 \rightarrow 0$.