

Transforms

1 Fast Fourier Transform

See section 12.2 in *Numerical Recipes*.

Notation

Data vector $\mathbf{x}$ with components $x_0, \dots, x_{N-1}$.

Discrete Fourier transform (the normalization differs from that in *Numerical Recipes*):

$$\tilde{x}_k = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} e^{2\pi i k n / N} x_n \quad k = 0, \dots, N-1 \quad (1)$$

In vector-matrix notation:

$$\tilde{\mathbf{x}} = \mathbf{M}\mathbf{x} \quad M_{kn} = \frac{1}{\sqrt{N}} e^{2\pi i k n / N}$$

Exercise A

Show that the transformation is unitary, that is

$$\mathbf{M}^\dagger \mathbf{M} = \mathbf{1},$$

where $\mathbf{1}$ denotes the $N \times N$ unit matrix and $\mathbf{M}^\dagger$ is the complex conjugate of the transpose of $\mathbf{M}$. What is the inverse of (1)?

Exercise B

Show that (assume N even)

$$\tilde{x}_k = \tilde{x}_k^e + e^{2\pi i k / N} \tilde{x}_k^o \quad k = 0, \dots, N-1$$

where

$$\tilde{x}_k^e = \frac{1}{\sqrt{N}} \sum_{m=0}^{N/2-1} e^{2\pi i k m / (N/2)} x_{2m} \quad \tilde{x}_k^o = \frac{1}{\sqrt{N}} \sum_{m=0}^{N/2-1} e^{2\pi i k m / (N/2)} x_{2m+1}$$

Note that $\tilde{x}_{k+N/2}^{e/o} = \tilde{x}_k^{e/o}$, so it is sufficient to calculate $\tilde{x}_k^e$ and $\tilde{x}_k^o$ for $k = 0, \dots, N/2 - 1$.

Exercise C

Estimate the number of operations needed to calculate the transform $\tilde{\mathbf{x}}$ if it is done as described in exercise B. What is the corresponding estimate for a direct calculation using (1)?

2 Wavelets

See section 13.10 in *Numerical Recipes*.

Notation

Data vector $\mathbf{x} = \mathbf{x}^{(0)}$ with components $x_0, \dots, x_{N-1}$ ($N = 2^k$).

Wavelet filter coefficients c_0, c_1, c_2, c_3 (fourth order Daubechies filter).

Filter matrix (zeros at empty places):

$$\mathbf{D}_m = \begin{bmatrix} c_0 & c_1 & c_2 & c_3 & & & & & & \\ c_3 & -c_2 & c_1 & -c_0 & & & & & & \\ & & c_0 & c_1 & c_2 & c_3 & & & & \\ & & c_3 & -c_2 & c_1 & -c_0 & & & & \\ \vdots & \vdots & & & & & \ddots & & & \\ & & & & & & & c_0 & c_1 & c_2 & c_3 \\ & & & & & & & c_3 & -c_2 & c_1 & -c_0 \\ & c_2 & c_3 & & & & & & c_0 & c_1 & \\ c_1 & -c_0 & & & & & & & c_3 & -c_2 & \end{bmatrix}$$

where the subscript m indicates that $\mathbf{D}_m$ is an $m \times m$ matrix. Let

$$\mathbf{F}_m = \left[\begin{array}{c|c} \mathbf{D}_m & \\ \hline & \mathbf{1}_{N-m} \end{array} \right]$$

where $\mathbf{1}_{N-m}$ is the $(N-m) \times (N-m)$ unit matrix.

Discrete wavelet transform:

$$\begin{aligned}
\mathbf{x}^{(0)} &\rightarrow \begin{bmatrix} x_0^{(1)} \\ w_0^{(1)} \\ x_1^{(1)} \\ w_1^{(1)} \\ \vdots \\ x_{N/2-1}^{(1)} \\ w_{N/2-1}^{(1)} \end{bmatrix} = \mathbf{F}_N \mathbf{x}^{(0)} \xrightarrow{\text{perm.}} \mathbf{x}^{(1)} = \begin{bmatrix} x_0^{(1)} \\ \vdots \\ x_{N/2-1}^{(1)} \\ w_0^{(1)} \\ \vdots \\ w_{N/2-1}^{(1)} \end{bmatrix} \\
&\rightarrow \begin{bmatrix} x_0^{(2)} \\ w_0^{(2)} \\ \vdots \\ x_{N/4-1}^{(2)} \\ w_{N/4-1}^{(2)} \\ w_0^{(1)} \\ \vdots \\ w_{N/2-1}^{(1)} \end{bmatrix} = \mathbf{F}_{N/2} \mathbf{x}^{(1)} \xrightarrow{\text{perm.}} \mathbf{x}^{(2)} = \begin{bmatrix} x_0^{(2)} \\ \vdots \\ x_{N/4-1}^{(2)} \\ w_0^{(2)} \\ \vdots \\ w_{N/4-1}^{(2)} \\ w_0^{(1)} \\ \vdots \\ w_{N/2-1}^{(1)} \end{bmatrix} \\
&\rightarrow \dots
\end{aligned}$$

so the full transformation is given by a matrix $\mathbf{M}$ of the form

$$\mathbf{M} = \mathbf{P}_{k-1} \mathbf{F}_4 \dots \mathbf{P}_2 \mathbf{F}_{N/2} \mathbf{P}_1 \mathbf{F}_N$$

where $\mathbf{P}_i$ are permutation matrices.

Exercise A

Show that $\mathbf{M}$ is orthogonal if $\mathbf{D}_m$ is orthogonal (for all even m), and that $\mathbf{D}_m$ is orthogonal if

$$c_0^2 + c_1^2 + c_2^2 + c_3^2 = 1 \quad (2)$$

$$c_2 c_0 + c_3 c_1 = 0 \quad (3)$$

What is the inverse of $\mathbf{M}$ if $\mathbf{D}_m$ is orthogonal?

Exercise B

Show that $c_3 x_n - c_2 x_{n+1} + c_1 x_{n+2} - c_0 x_{n+3} = 0$ for any linear signal $x_n = \alpha n + \beta$ if

$$c_3 - c_2 + c_1 - c_0 = 0 \quad (4)$$

$$-c_2 + 2c_1 - 3c_0 = 0 \quad (5)$$

Exercise C

Verify that

$$c_0 = (1 + \sqrt{3})/4\sqrt{2} \quad c_1 = (3 + \sqrt{3})/4\sqrt{2} \quad c_2 = (3 - \sqrt{3})/4\sqrt{2} \quad c_3 = (1 - \sqrt{3})/4\sqrt{2}$$

is a solution to the four equations in exercises A and B.

3 Principal Components Analysis

Notation

Measurement vector $\mathbf{x}$ with components $x_1, \dots, x_N$ (which may have no particular temporal or spatial ordering). Each x_i is viewed as a random number.

Covariance matrix $\mathbf{C}^x$ defined by $C_{ij}^x = \langle (x_i - \langle x_i \rangle)(x_j - \langle x_j \rangle) \rangle$.

Exercise A

Suppose that $\mathbf{y} = \mathbf{M}\mathbf{x}$, where $\mathbf{M}$ is an arbitrary orthogonal matrix. Show that the covariance matrix $\mathbf{C}^y$, $C_{ij}^y = \langle (y_i - \langle y_i \rangle)(y_j - \langle y_j \rangle) \rangle$, can be written as

$$\mathbf{C}^y = \mathbf{M}\mathbf{C}^x\mathbf{M}^T.$$

Because $\mathbf{C}^x$ is a real symmetric matrix, it follows that the matrix $\mathbf{M}$ can be chosen so that $\mathbf{C}^y$ becomes diagonal. The components y_i are then uncorrelated.

Exercise B

Consider a measurement of a three-component vector $\mathbf{x}$, where x_1 and x_2 are independent random numbers with zero mean and unit variance and $x_3 = x_1 + x_2$. Calculate the covariance matrix $\mathbf{C}^x$ and its eigenvalues.

4 Mellin transform

In particle physics it is not uncommon to have evolution equations eg. describing the probability of finding a quark or a gluon inside a proton when probed at some resolution scale t , where the quark or gluon carries some fraction x of the proton energy. These

probabilities cannot be calculated from first principles but if they are known at some resolution scale we can calculate it at a higher scale. If we increase the resolution we may find that what we saw as a quark at the lower scale was actually a quark which had radiated a gluon retaining only a fraction z of its energy. If $q(x, t)$ is the quark *density* and $P(x)$ the *splitting fuction* giving the probability of a quark radiating a gluon retaining a fraction z of its energy, the evolution can be written as a integro-differential equation on the form:

$$\frac{dq(x, t)}{dt} = \int_x^1 \frac{d\xi}{\xi} P(x/\xi) q(\xi, t)$$

The Mellin transform is given by

$$\tilde{f}_\omega = \int_0^1 x^{\omega-1} f(x) dx$$

and its inverse

$$f(x) = \frac{1}{2\pi i} \int_C d\omega x^{-\omega} \tilde{f}_\omega$$

where the integral is along a contour C in the complex ω plane which is parallel to the imaginary axis and to the right of all singularities. $\tilde{f}_\omega$ are often refered to as the *moments* of $f(x)$

Taking the moments on each side of the evolution equation we get:

$$\int_0^1 dx x^{\omega-1} \frac{dq(x, t)}{dt} = \frac{d\tilde{q}_\omega(t)}{dt}$$

and

$$\int_0^1 dx x^{\omega-1} \int_x^1 \frac{d\xi}{\xi} P(x/\xi) q(\xi, t) = \int_0^1 \frac{d\xi}{\xi} \int_0^\xi \frac{dx}{x} x^\omega P(x/\xi) q(\xi, t)$$

making the substitution $x = z\xi$ this gives

$$\int_0^1 \frac{d\xi}{\xi} \xi^\omega q(\xi, t) \int_0^1 \frac{dz}{z} z^\omega P(z) = \tilde{q}_\omega(t) \cdot \tilde{P}_\omega$$

and we have obtained a set of simple ODE:

$$\frac{d\tilde{q}_\omega(t)}{dt} = \tilde{q}_\omega(t) \cdot \tilde{P}_\omega$$

Note, however, that the inverse Mellin transform may not be trivial to perform.