

Field Theory

The material here is spread over a few places in the book. It is to be read alongside the book, it presents the ideas of field theory with a slightly different emphasis.

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1 Introduction

The purpose of this note is to give a bit of a feeling for field theory and the aspects of it that are needed for understanding the rest of the course.

Obviously a two hour lecture is not a substitute for a full course on field theory. The ideas that should be taken home from this lecture are

- The concept of a field.
- The field represents particle quanta as seen in the expansion in creation and annihilation operators.
- Particles and anti-particles.
- Symmetries lead to Conserved Currents and thus to conserved quantities, and some idea how the proof of Noether's theorem goes.
- Propagation from a source to another source/sink and how it is related to a potential.
- Terms in the Lagrangians with more than two fields present lead to interactions via vertices.
- The simplified version of Feynman rules.
- The kinetic terms for the different types of fields, including the factors of $1/2$ when identifying the mass of a particle.
- In particular you do *not* need to understand Lagrangian(-density)s and how they lead to equations of motion, accepting that the Euler-Lagrange equations is the result is more than sufficient. A particle physicist will in general just take the relevant Lagrangian (density) and use immediately the Feynman rules that follow from it.

2 Some relativistic notation

This can be found in Sect.2.1 in the book.

For vector indices: i, j, k, \dots run over 1,2,3 or x, y, z and $\mu, \nu, \alpha, \beta, \dots$ run over 0,1,2,3 or t, x, y, z .

The metric $g_{\mu\nu}$ and the inverse metric $g^{\mu\nu}$ are in matrix notation (first index row, second index column, elements that are blank are zero)

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \quad (1)$$

These are used to raise and lower indices, i.e. make them an upper, superscript or lower, subscript index. As an example, take a four vector $p = (p^0, p^1, p^2, p^3) = (E, \vec{p})$ and a four vector q then we get:

$$p_\mu = g_{\mu\nu} p^\nu \quad p^\mu = g^{\mu\nu} p_\nu \quad g_{\mu\nu} = g_{\mu\alpha} g_{\nu\beta} g^{\alpha\beta} \quad p \cdot q = p^\mu q^\nu g_{\mu\nu} = p_\mu q^\mu = p^\mu q_\mu, \quad (2)$$

where as (almost) always repeated indices are summed over.

3 Continuum and relativistic Lagrangians

Note this is no derivation, just a statement of facts, consult a book on classical mechanics for a derivation. Some related material is in App. F in the book.

For a single particle with mass m and position \vec{x} moving in the force created by a potential $V(\vec{x})$, Newton's equation is $m \frac{d^2}{dt^2} \vec{x} = -\vec{\nabla} V$. An alternative way to describe classical mechanics is to use the Lagrangian method. We introduce the Lagrangian

$$L(\vec{x}) = \frac{m}{2} \vec{v}^2 - V(\vec{x}) = \frac{m}{2} \left(\frac{d\vec{x}}{dt} \right)^2 - V(\vec{x}). \quad (3)$$

We now write the action as $S = \int dt L(\vec{x}(t))$, i.e. we think of all three coordinate as functions of time $x^i(t)$. Minimizing the action S leads to the equation for each value of $i = 1, 2, 3$

$$\frac{d}{dt} \frac{\delta L}{\delta \left(\frac{dx^i(t)}{dt} \right)} - \frac{\delta L}{\delta x^i(t)} = 0. \quad (4)$$

This is called the Euler-Lagrange equation. The overall time derivative should be understood as acting on everywhere a time appears, also as argument of $\vec{x}(t)$. The derivatives denoted by δ are taking the derivative like a partial derivative¹.

If we have a system of n -particles, the Lagrangian becomes

$$L = \sum_{k=1, n} \left(\frac{dx^i(k)}{dt} \right)^2 - V(\vec{x}_{(1)}, \dots, \vec{x}_{(n)}). \quad (5)$$

¹A simple way to see this is to think of the δ -derivative (also called functional derivative) as: $\frac{\delta}{\delta f(u)} \int dv f(v) g(v) \equiv g(u)$. This is for the example of functions of one variable but we will use the generalization to functions of all spacetime coordinates.

and we get $3n$ Euler-Lagrange equations from the minimization of the action $S = \int dt L(\vec{x}(t), d\vec{x}(t)/dt)$.

$$\frac{d}{dt} \frac{\delta L}{\delta \left(\frac{dx^i_{(k)}}{dt} \right)} - \frac{\delta L}{\delta x^i_{(k)}} = 0. \quad (6)$$

This generalizes to continuum variables, the $x^i_{(k)}(t)$ are replaced by quantities in each space point $\phi(\vec{x}, t)$ where the \vec{x} is now like the k in the n -particle case and the \sum_k becomes replaced by an integral over all of space. We denote it as $\phi(x)$ where x is the space-time coordinate. The Lagrangian then becomes $L = \int d^3x \mathcal{L}(\partial_\mu \phi, \phi)$ where also spatial derivatives can occur (see e.g. the derivation of the equation of motion for a stretched string). \mathcal{L} is the Lagrangian density, but usually simply called the Lagrangian by particle physicists. Minimizing the action $S = \int dt L = \int d^4x \mathcal{L}(\phi(x), \partial_\mu \phi(x))$ leads to the Euler-Lagrange equation

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi)} - \frac{\delta \mathcal{L}}{\delta \phi} = 0. \quad (7)$$

The partial derivative $\partial_\mu = \partial/\partial x^\mu$ acts also on the x^μ dependence inside $\phi(x)$. The derivatives denoted by delta are like partial derivatives w.r.t. $\phi(x)$ in the way that $\frac{\delta}{\delta \phi(x)} \int d^4y \phi(y) f(y) = f(x)$, called functional derivatives.

4 Lagrangians for electromagnetism

This can be partly found in Ch. 2.2 of the book.

An example of a field you all have encountered before is the electromagnetic field. The electric $\vec{E}(\vec{x}, t)$ and magnetic $\vec{B}(\vec{x}, t)$ are an example of a (three-)vector quantity that takes on different values at each space time point.

With units such that $c = \varepsilon_0 = \mu_0 = 1$ Maxwell's equations are with \vec{E} the electric field, \vec{B} the magnetic field, ρ the charge density and \vec{J} the current density:

$$\begin{aligned} \nabla \cdot \vec{E} &= \rho & \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \nabla \cdot \vec{B} &= 0 & \nabla \times \vec{B} &= \vec{J} + \frac{\partial \vec{E}}{\partial t} \end{aligned} \quad (8)$$

Lorentz discovered that these had a symmetry, the Lorentz transformation which mixes the electric and magnetic fields. The nicest way to put them in an object that behaves simply under Lorentz transformations is to introduce the antisymmetric field strength tensor $F_{\mu\nu} = -F_{\nu\mu}$ with (note upper indices 1, 2, 3 = x, y, z not powers)²

$$\begin{aligned} F^{0i} &= -E^i & F^{ij} &= -\epsilon^{ijk} B^k \\ F^{\mu\nu} &= \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix} \end{aligned} \quad (9)$$

²The book has a different sign for B^i , probably forgot that $\nabla_i = -\partial^i$ for $i = x, y, z$.

The object ϵ^{ijk} is defined as $\epsilon^{123} = 1$ and it is antisymmetric when you interchange two indices. A similar object exists with four indices $\epsilon^{\mu\nu\alpha\beta}$ with $\epsilon^{0123} = 1$ and antisymmetric under the interchange of two indices. These are called Levi-Civita tensors.

In this notation Maxwell's equations become

$$\epsilon^{\mu\nu\alpha\beta} \partial_\nu F_{\alpha\beta} = 0, \quad \partial_\mu F^{\mu\nu} = J^\nu. \quad (10)$$

The fourvector current is $J = (\rho, \vec{J})$ and it transforms under Lorentz transformations as all four-vectors (i.e. like x^μ).

The two Maxwell's equations not involving ρ or \vec{j} can be solved by introducing a potential V and a vector potential \vec{A} with

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \nabla \times \vec{A}. \quad (11)$$

This can be written in a nice relativistic form as

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \quad (12)$$

A Lagrangian density (with A_μ as the degree of freedom) that leads to electromagnetism is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J^\mu A_\mu \quad (13)$$

Note that the current is conserved, we have

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0 \quad \Rightarrow \quad \partial_\mu J^\mu = 0. \quad (14)$$

The first equation says that if the charge density changes it has to flow somewhere³ where the last form shows the relativistic variant of the conservation or continuity equation of charge.

5 Lagrangians in Particle Physics

Lagrangian densities are not the only way to specify a field theory but they have become the conventional way of doing it. The reasons are many but the main reason is that it is very easy to see whether a theory is invariant or not under various symmetries. In particular, the Lagrangian formulation is manifestly Lorentz invariant as soon as the Lagrangian is Lorentz invariant. This does not play much of a role at the level of the course but is very important when doing full fledged field theory.

³Using Gauss's theorem this can be rewritten as the change of the integral over a volume of ρ is equal to the flux of \vec{J} through the volume's surface.

6 Real scalar field

See Chapter 4.1 in the book for some related material.

This is the simplest example of a field theory. We take as degree of freedom a single real number in each point denoted by $\phi(x)$. If this corresponds to a particle, the resulting wave-solutions should be such that $m^2 = E^2 - \vec{p}^2$ or

$$\left(\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + m^2 \right) \phi(x) = 0 \quad (15)$$

when replacing E by the time derivative and \vec{p} by spatial derivatives. In the notation above this is

$$(\partial^2 + m^2) \phi(x) = 0 \quad (16)$$

known as the Klein-Gordon equation. In terms of momenta \vec{p} and energies $E = \pm \sqrt{\vec{p}^2 + m^2}$ and $p = E, \vec{p}$ solutions are plane waves

$$\phi(x) \propto e^{-ip \cdot x}. \quad (17)$$

Deriving the correct normalization factor involves normalizing the states to one-particle per unit volume, we will just give you the correct factors later needed for Feynman diagrams.

This system has solutions with energies $-\infty \leq E \leq -m$ and $m \leq E \leq \infty$. A wave equation as (16) interpreted as a wave-function for a single particle has a number of deficiencies that are all more or less related to the existence of negative energy eigenstates of arbitrarily negative size. The system as described above has no ground state.

The solution is to not interpret $\phi(x)$ as a wave-function of a single particle but instead take $\phi(x)$ itself as the degrees of freedom of the system and then quantize it. This means an enormous increase in the degrees of freedom from 3 (the three spatial coordinates of a single particle) to a real number at every space-point, and the number can then vary with time.

The negative energy problem is solved by reinterpreting the modes with negative energy in a way similar to the Dirac sea, for those of you having seen that. A mode with energy E has time dependence e^{-iEt} . We define $E = \sqrt{\vec{p}^2 + m^2}$ and $p = (E, \vec{p})$ and expand the field in Fourier modes via

$$\phi(x) \propto \int \frac{d^3p}{(2\pi)^3} \left(a_{\vec{p}} e^{-ip \cdot x} + a_{\vec{p}}^\dagger e^{ip \cdot x} \right). \quad (18)$$

The second term contains the negative energy solutions. The field is real since the second term is the complex conjugate of the first. We now go over to quantum field theory and define $a_{\vec{p}}$ and $a_{\vec{p}}^\dagger$ to be operators (called annihilation and creation operators respectively) and define the vacuum $|0\rangle$ to be the state satisfying

$$a_{\vec{p}}|0\rangle = 0. \quad (19)$$

(This step corresponds to thinking of the negative energy states as filled in the Dirac sea picture).

A Lagrangian that from its Euler-Lagrange equation (7) produces the Klein-Gordon equation (16) is

$$\mathcal{L}_r = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2. \quad (20)$$

7 Complex scalar field

Ch. 2.6 in the book. We can now take two real scalar field ϕ_1 and ϕ_2 with same mass m and we get the Lagrangian (20) twice

$$\mathcal{L}_c = \frac{1}{2} (\partial_\mu \phi_1 \partial^\mu \phi_1 + \partial_\mu \phi_2 \partial^\mu \phi_2) - \frac{1}{2} m^2 (\phi_1^2 + \phi_2^2). \quad (21)$$

This can be written in terms of single *complex* field $\Phi = (\phi_1 + i\phi_2)/\sqrt{2}$ as

$$\mathcal{L}_c = \partial_\mu \Phi^* \partial^\mu \Phi - m^2 \Phi^* \Phi. \quad (22)$$

Getting the Euler-Lagrange equation from a complex Lagrangian can be done in two ways, either by looking at the real and imaginary part and treating those as separate entities, or as treating the field and its complex conjugate as independent entities and using (7) with derivatives w.r.t. Φ and Φ^* . Taking the derivative w.r.t. Φ leads to

$$(\partial^2 + m^2) \Phi^* = 0 \quad (23)$$

and with Φ to

$$(\partial^2 + m^2) \Phi = 0 \quad (24)$$

Both Φ and Φ^* thus describe particles with mass m .

7.1 Antiparticles

As done earlier for the real field we now interpret Φ also as a quantum field. Expanding in annihilation and creation operators can be done again

$$\Phi(x) \propto \int \frac{d^3 p}{(2\pi)^3} \left(b_{\vec{p}} e^{-ip \cdot x} + c_{\vec{p}}^\dagger e^{ip \cdot x} \right). \quad (25)$$

Notice that since Φ is now complex the two terms do not need to be each other's Hermitian conjugate. We interpret $b_{\vec{p}}$ as the annihilation operator for a particle and $c_{\vec{p}}^\dagger$ as the creation operator for an *antiparticle*. The vacuum is defined as

$$b_{\vec{p}}|0\rangle = c_{\vec{p}}|0\rangle = 0. \quad (26)$$

The Hermitian conjugate field has the expansion

$$\Phi^*(x) \propto \int \frac{d^3 p}{(2\pi)^3} \left(c_{\vec{p}} e^{-ip \cdot x} + b_{\vec{p}}^\dagger e^{ip \cdot x} \right). \quad (27)$$

Φ^* contains annihilation operators for an antiparticle and creation operators for a particle.

7.2 Symmetries

Writing the Lagrangian of two equal mass real scalar fields in terms of a single complex scalar field has a bonus. (22) has a symmetry that was not obvious in the other formulation. Doing a global symmetry transformation with a constant χ

$$\Phi(x) \rightarrow \Phi'(x) = e^{i\chi}\Phi(x), \quad (28)$$

and the complex conjugate of this for Φ^* , leaves \mathcal{L}_c invariant, i.e. $\mathcal{L}_c(\Phi', \Phi'^*) = \mathcal{L}_c(\Phi, \Phi^*)$.

8 Noether's theorem

I show it for the complex scalar field, see Ch. 2.6.

Take the Lagrangian for the complex scalar field. It does not transform under the symmetry operation (28). We will now show by doing a small transformation (treat χ as infinitesimal or $\delta\Phi = i\chi\Phi$ and $\delta\partial_\mu\Phi = i\chi\partial_\mu\Phi$, since χ does not depend on x). We write the change $\mathcal{L} = 0$ in a more complicated way

$$\begin{aligned} 0 &= \frac{\delta\mathcal{L}}{\delta\Phi}\delta\Phi + \frac{\delta\mathcal{L}}{\delta\partial_\mu\Phi}\delta\partial_\mu\Phi + \frac{\delta\mathcal{L}}{\delta\Phi^*}\delta\Phi^* + \frac{\delta\mathcal{L}}{\delta\partial_\mu\Phi^*}\delta\partial_\mu\Phi^* \\ &= i\chi \left[\frac{\delta\mathcal{L}}{\delta\Phi}\Phi + \frac{\delta\mathcal{L}}{\delta\partial_\mu\Phi}\partial_\mu\Phi - \frac{\delta\mathcal{L}}{\delta\Phi^*}\Phi^* - \frac{\delta\mathcal{L}}{\delta\partial_\mu\Phi^*}\partial_\mu\Phi^* \right] \\ &= i\chi \left[\partial_\mu \frac{\delta\mathcal{L}}{\delta\partial_\mu\Phi}\Phi + \frac{\delta\mathcal{L}}{\delta\partial_\mu\Phi}\partial_\mu\Phi - \partial_\mu \frac{\delta\mathcal{L}}{\delta\partial_\mu\Phi^*}\Phi^* - \frac{\delta\mathcal{L}}{\delta\partial_\mu\Phi^*}\partial_\mu\Phi^* \right] \\ &= i\chi\partial_\mu \left[\frac{\delta\mathcal{L}}{\delta\partial_\mu\Phi}\Phi - \frac{\delta\mathcal{L}}{\delta\partial_\mu\Phi^*}\Phi^* \right] \end{aligned} \quad (29)$$

We used the Euler-Lagrange equations in going from the second to the third line. The last line shows we now have a conserved current⁴

$$J^\mu = i \left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\Phi}\Phi - \frac{\delta\mathcal{L}}{\delta\partial_\mu\Phi^*}\Phi^* \right). \quad (30)$$

For the complex scalar the actual current is

$$J^\mu = i (\Phi\partial^\mu\Phi^* - \Phi^*\partial^\mu\Phi). \quad (31)$$

As mentioned above it is real, if we interchange Φ and Φ^* it changes sign. All charges for antiparticles have thus the opposite sign than for anti-particles. We will later interpret the electromagnetic current in this way⁵.

⁴The i is included to have a real/Hermitian current.

⁵If we have two different fields Φ_1 and Φ_2 they can be given different q_1 and q_2 by generalizing the transformations to $\Phi_1 \rightarrow e^{iq_1\chi}\Phi_1$ and $\Phi_2 \rightarrow e^{iq_2\chi}\Phi_2$

That symmetries lead to conserved quantities is a very general conclusion, we have simply shown it for one case. A few other examples are:

Symmetry	Conserved quantity
Translation	momentum
Translation in time	Energy
Rotation	Angular momentum
Phase invariance	charge

The phase invariance is what leads to an interaction like electromagnetism, the changes in the phase then come extra with a factor of the charge of the particle.

9 The propagator and interactions

An argument for Feynman diagrams can be found in Ch. 2.7. We use the example of a real scalar field.

We add a source term to the Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 + \rho(x) \phi(x). \quad (32)$$

The Euler-Lagrange equation is now with a source on the right hand side.

$$\partial^2 \phi + m^2 \phi = \rho. \quad (33)$$

We now choose a time-independent source $\rho(x) = g\delta^3(\vec{x})$ and assume time-independent solutions for $\phi(x)$. This can be solved using Fourier transformation leading to

$$\phi(\vec{x}) = g \int \frac{d^3 p}{(2\pi)^3} \frac{e^{i\vec{p}\cdot\vec{x}}}{\vec{p}^2 + m^2} = \frac{g}{4\pi} \frac{e^{-mr}}{r}, \quad (34)$$

with $r = |\vec{x}|$.

Calculating the Hamiltonian from (32) and putting a test charge g at position \vec{y} via adding $\rho = g\delta^3(\vec{x} - \vec{y})$, we get a change in energy that we can interpret as coming from a potential times the charge

$$gV(|\vec{y}|) = -\frac{g^2}{4\pi} \frac{e^{-m|\vec{y}|}}{|\vec{y}|}. \quad (35)$$

We can now interpret this instead as exchange of a ϕ -particle between two sources, we get a factor⁶ of ig , the coupling constant, for each source and a factor of (the four-vector p now)

$$\frac{i}{p^2 - m^2} \quad (36)$$

for the exchanged particle. The last factor is called the propagator.

⁶I now include all factors of i that a correct derivation of Feynman diagrams would give and the generalization to time-dependence.

One more generalization is needed, if we add terms with three or more fields to the Lagrangian we treat these like the source terms. The other fields in those terms can thus act as sources for the field that conveys the information or propagates. See below how this works in practice for an example calculation.

10 Feynman diagrams

Very shortly in Ch. 2.9 in the book.

10.1 What are now Feynman diagrams?

Feynman diagrams are a technique to solve quantum field theory. Their main use is to calculate the amplitude (or rather i times the amplitude) for a state with specified incoming particles with momenta and spins specified to evolve to a different state with specified particles and their momenta and spins.⁷

We divide the Lagrangian into

- Kinetic terms: those with two fields as described earlier. These terms produce the propagators and give the lines that connect different points of a Feynman diagram. Internal lines must be summed over all possible momenta and spins.
- Interaction terms: those terms with three or more fields. This part is usually called \mathcal{L}_I . It provides connection points called vertices where three or more lines meet.
- At the vertices momentum is conserved: the sum over all incoming momenta must be equal to the sum over all outgoing momenta at each vertex. You can check that for tree level diagrams, i.e. no closed loops, this means that all occurring momenta are specified in terms of the incoming and outgoing momenta.
- The previous point also leads to momentum conservation for the full diagram. The sum over all outgoing momenta is equal to the sum over all incoming momenta.

To construct the amplitudes for a process one now does the following

1. Connect via vertices and internal lines all incoming particles to all outgoing particles.
2. Internal lines are propagators. These can be seen as a particle traveling in one direction and as an anti-particle traveling in the opposite direction. Both terms contribute but the propagator automatically includes both. Simply take one of the two cases, you can check that both give the same result. For an internal line corresponding to a scalar particle with mass m and momentum k the propagator is $i/(k^2 - m^2)$

⁷It is sufficient if all momenta are different, this requirement is there to have contributions from connected diagrams only.

3. Each vertex gives a factor i and all extra parts that appear in the term in \mathcal{L}_I after the creation operators have been used to create the particles corresponding to the lines leaving the vertex and the annihilation operators have been used remove the particles from the lines coming in to the vertex. The exponential factors are responsible for the momentum conservation and are take into account that way.

In the next section, an example is worked out in more detail to make it somewhat clearer.

10.2 An example worked out in detail

We look at a Lagrangian with a real scalar field ϕ and a complex scalar field Ψ with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \partial_\mu\Psi^*\partial^\mu\Psi - M^2\Psi^*\Psi + g\phi\Psi^*\Psi + \lambda\phi^3$$

The first four terms are the kinetic terms and give:

- internal ϕ line with momentum k : propagator $i/(k^2 - m^2)$.
- internal Ψ line with momentum k : propagator $i/(k^2 - M^2)$.

Let us now look at the next-to-last term. For ϕ use (18) and for Ψ use (25). So $a_{\vec{p}}$ annihilates a ϕ -particle, $b_{\vec{p}}$ annihilates a Ψ -particle and $c_{\vec{p}}$ annihilates an anti- Ψ particle. The daggered versions create instead.

We can now write out the interaction term $g\phi\Psi^*\Psi$ in glorious detail

$$\begin{aligned} g\phi\Psi^*\Psi = \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \frac{d^3p_3}{(2\pi)^3} \left\{ \right. & \quad (37) \\ & ga_{\vec{p}_1} c_{\vec{p}_2} b_{\vec{p}_3} e^{i(-p_1-p_2-p_3)\cdot x} + ga_{\vec{p}_1} c_{\vec{p}_2} c_{\vec{p}_3}^\dagger e^{i(-p_1-p_2+p_3)\cdot x} \\ & + ga_{\vec{p}_1} b_{\vec{p}_2}^\dagger b_{\vec{p}_3} e^{i(-p_1+p_2-p_3)\cdot x} + ga_{\vec{p}_1} b_{\vec{p}_2}^\dagger c_{\vec{p}_3}^\dagger e^{i(-p_1+p_2+p_3)\cdot x} \\ & + ga_{\vec{p}_1}^\dagger c_{\vec{p}_2} b_{\vec{p}_3} e^{i(p_1-p_2-p_3)\cdot x} + ga_{\vec{p}_1}^\dagger c_{\vec{p}_2} c_{\vec{p}_3}^\dagger e^{i(p_1-p_2+p_3)\cdot x} \\ & \left. + ga_{\vec{p}_1}^\dagger b_{\vec{p}_2}^\dagger b_{\vec{p}_3} e^{i(p_1+p_2-p_3)\cdot x} + ga_{\vec{p}_1}^\dagger b_{\vec{p}_2}^\dagger c_{\vec{p}_3}^\dagger e^{i(p_1+p_2+p_3)\cdot x} \right\}. \end{aligned}$$

The sum over the three momenta is there for all eight terms. Let us now analyze the contents of these eight terms and what they do. I label the terms case 1, ..., 8. A ϕ -particle with momentum p_1 is denoted by $\phi(p_1)$, a Ψ -particle with momentum p_2 is denoted by $\Psi(p_2)$, and a Ψ -anti-particle with momentum p_3 is denoted by $\Psi^*(p_3)$. The eight terms give different in/out combinations at the vertex with somewhat different factors and momentum conservation. The rules of the previous section lead to the eight cases (one from each term)

Case	in/out	factor in diagram	momentum conservation
1	$\phi(p_1)$ in, $\Psi^*(p_2)$ in, $\Psi(p_3)$ in	ig	$p_1 + p_2 + p_3 = 0$
2	$\phi(p_1)$ in, $\Psi^*(p_2)$ in, $\Psi^*(p_3)$ out	ig	$p_1 + p_2 = p_3$
3	$\phi(p_1)$ in, $\Psi(p_2)$ out, $\Psi(p_3)$ in	ig	$p_1 + p_3 = p_2$
4	$\phi(p_1)$ in, $\Psi(p_2)$ out, $\Psi^*(p_3)$ out	ig	$p_1 = p_2 + p_3$
5	$\phi(p_1)$ out, $\Psi^*(p_2)$ in, $\Psi(p_3)$ in	ig	$p_2 + p_3 = p_1$
6	$\phi(p_1)$ out, $\Psi^*(p_2)$ in, $\Psi^*(p_3)$ out	ig	$p_2 = p_1 + p_3$
7	$\phi(p_1)$ out, $\Psi(p_2)$ out, $\Psi(p_3)$ in	ig	$p_3 = p_1 + p_2$
8	$\phi(p_1)$ out, $\Psi(p_2)$ out, $\Psi^*(p_3)$ out	ig	$0 = p_1 + p_2 + p_3$

The other interaction term can be expanded analogously

$$\lambda\phi^3 = \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \frac{d^3p_3}{(2\pi)^3} \left\{ a_{\vec{p}_1} a_{\vec{p}_2} a_{\vec{p}_3} e^{i(-p_1-p_2-p_3)\cdot x} + \dots \right\}$$

and a similar table can be constructed for this term as well, the complication that the same field shows up several times is discussed below.

The factor can be remembered by: an i and the remaining factors are what was present as constants in the Lagrangian.

Let us now look at the process $\phi(q_1)$ and $\psi(q_2)$ in, $\phi(q_3)$ and $\psi(q_4)$ out.

We denote a ϕ line by a dashed line and Ψ by a full line:

ϕ : - - - - -

Ψ : _____

The process we want to get at is

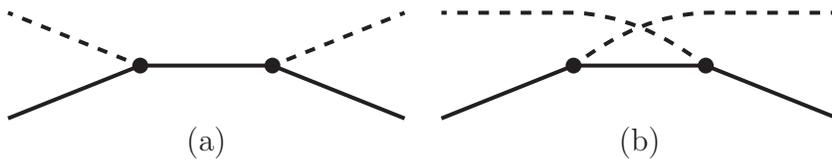


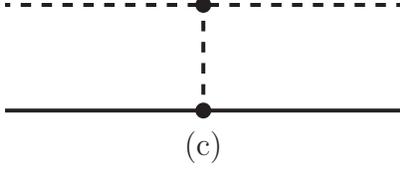
We should now replace the $\dots\dots\dots$ by *all* combinations of vertices and lines that connect the incoming lines on the left to the outgoing lines on the right.

The Lagrangian has two possible vertices from the two interaction terms in the Lagrangian



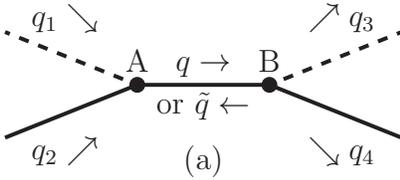
After some thinking you should find three ways (without loops) to connect the incoming lines to the outgoing lines





In diagram (b) note that the dashed lines cross but there is no vertex there.

Let us now work out diagram (a) in detail. Let's write down the momenta again



The vertex A is case 3 from the table with $p_1 = q_1$, $p_3 = q_2$ and $p_2 = q$ so we get that $q = q_1 + q_2$ and a factor (ig) . Note that if we had chosen the \tilde{q} option instead we would have used case 1 and $\tilde{q} = -q_1 - q_2$ and obtained the same factor.

The vertex B is case 7 with $p_1 = q_3$, $p_2 = q_4$ and $p_3 = q$ so $q = q_3 + q_4$ and we obtain a factor (ig) . Note that the two requirements on q require overall momentum conservation $q_3 + q_4 = q_1 + q_2$.

The internal line is a Ψ line with momentum q contributing the propagator $i/(q^2 - M^2)$. The Feynman amplitude $i\mathcal{A}$ for this diagram is thus by putting together all factors

$$i\mathcal{A}^{(a)} = \frac{i^3 g^2}{(q_1 + q_2)^2 - M^2}.$$

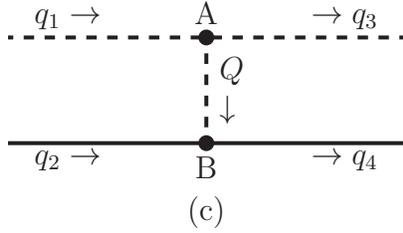
You can check that if you had chosen \tilde{q} instead for the internal line the final expression would have been identical. The beauty of Feynman's derivation is that the two cases are nicely combined in one expression for the propagator.

Diagram (b) can be treated in the same way with the internal line having momentum $k = q_2 - q_3$ from the left vertex and $k = q_4 - q_1$ from the right vertex, we thus again get overall momentum conservation. You can check that the full expression for this diagram is

$$i\mathcal{A}^{(b)} = \frac{i^3 g^2}{(q_2 - q_3)^2 - M^2},$$

when working out all the factors similar to what was done for diagram(a).

Diagram (c) has an additional complication, since the ϕ field appears several times in the vertex A on top of the diagram.



This leads to extra terms in the expression. Let us write the term $\lambda\phi^3 = \lambda\phi_1\phi_2\phi_3$ and do the expansion in creation and annihilation operators with ϕ_1 having momentum p_1 , ϕ_2 p_2 and ϕ_3 p_3 . Then of the eight cases you get when writing out the vertex three cases can contribute but each of them does it twice. Thus we get

$\phi(q_1)$ in annihilated by	$\phi(q_3)$ out created by	$\phi(Q)$ out created by	factor
ϕ_1 $p_1 = q_1$	ϕ_2 $p_2 = q_3$	ϕ_3 $p_3 = Q$	$i\lambda$
ϕ_1 $p_1 = q_1$	ϕ_3 $p_3 = q_3$	ϕ_2 $p_2 = Q$	$i\lambda$
ϕ_2 $p_2 = q_1$	ϕ_1 $p_1 = q_3$	ϕ_3 $p_3 = Q$	$i\lambda$
ϕ_2 $p_2 = q_1$	ϕ_3 $p_3 = q_3$	ϕ_1 $p_1 = Q$	$i\lambda$
ϕ_3 $p_3 = q_1$	ϕ_1 $p_1 = q_3$	ϕ_2 $p_2 = Q$	$i\lambda$
ϕ_4 $p_3 = q_1$	ϕ_2 $p_2 = q_3$	ϕ_1 $p_1 = Q$	$i\lambda$

The factor is always the same, this would not have been the case if there had been derivatives. If you check how the momentum conservation goes for all the options, you will see they always lead to $Q = q_1 - q_3$. The upper vertex is the sum of all the six contributions in the table and gives thus a factor $6i\lambda$.

The propagator is ϕ propagator and gives $i/(Q^2 - m^2)$ and the bottom vertex gives, using the same reasonings as used for diagram (a), ig . The final expression for the amplitude from diagram (c) is thus

$$i\mathcal{A}^{(c)} = \frac{i^3 6g\lambda}{(q_1 - q_3)^2 - m^2}$$

The full tree level Feynman amplitude for this process is the sum of the three terms

$$i\mathcal{A}^{tree} = i\mathcal{A}^{(a)} + i\mathcal{A}^{(b)} + i\mathcal{A}^{(c)}.$$

11 Fermions and vector fields

A vector field (spin one) is described by a real or complex four-vector field. There are thus four fields and they are described by a polarization vector ε_μ which depends on the spin state (or polarization) and the momentum. For fermions we have a Dirac Spinor instead, with spinor solutions.

11.1 Extra complications in Feynman diagrams for spins, fermions and other degrees of freedom

- For spins, these are dealt with by spinors and polarization vectors.
- The expressions for the vertices get extra factors of $u_{\vec{p},s}$, $v_{\vec{p},s}$, $\bar{u}_{\vec{p},s}$, $\bar{v}_{\vec{p},s}$, $\varepsilon_{\vec{p},s\mu}$ and/or $\varepsilon_{\vec{p},s\mu}^*$.
- Remember also to keep in the expressions for the vertex all the other parts, e.g. gamma matrices, and the indices that need to be summed over.
- In internal lines always sums appear over the spinors or polarization vectors that appear on both sides.

- For vectors this leads to the polarization sum $\sum_s \varepsilon_{\vec{p},s\mu}^* \varepsilon_{\vec{p},s\nu} = -g_{\mu\nu}$

In general there are extra terms but these complications do not appear at this level.

- For fermions in internal lines, always the combination (because both particles and antiparticles contribute)

$$\sum_s (u_{\vec{p},s} \bar{u}_{\vec{p},s} + v_{-\vec{p},s} \bar{v}_{-\vec{p},s}) = p_\mu \gamma^\mu + m,$$

appears and the equal sign follows from the properties of solutions of the Dirac equation. This way, you will always connect all spinors together in consistent combinations when you connect all vertices, keeping track of Dirac indices.

- One more complication appears for fermions. If the vertices are written in the form $\bar{\psi}_1 \Gamma \psi_2$ then the rules are
 - an extra -1 for each closed fermion loop
 - an extra (-1) for each crossing of fermion lines without a vertex.

It is the last sign that corresponds to the minus sign in the wave function when interchanging two fermions.

- If additional indices appear in the vertices, they can be dealt with similarly by keeping track of them and having the propagator diagonal in these indices. Colour connections through diagrams can be dealt with in this way.

In this course we will use order of magnitude arguments and dimensional analysis for all of these extra factors.

12 The kinetic and mass terms for the fields we use

The kinetic terms and the mass terms, those containing exactly two fields, are assumed to be diagonalized. This means that all terms with two fields are of one of the forms, for a field corresponding to a particle with mass m

Field	Lagrangian	Field strength
real scalar field ϕ	$\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$	
complex scalar field Φ	$\partial_\mu\Phi^*\partial^\mu\Phi - m^2\Phi^*\Phi$	
Dirac fermion ψ	$\bar{\psi}i\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi$	
real vector field A_μ	$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2A_\mu A^\mu$	$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$
complex vector field W_μ	$-\frac{1}{2}W_{\mu\nu}^*W^{\mu\nu} + m^2W_\mu^*W^\mu$	$W_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu$

Note the factors of 1/2 and the signs.