Problem 1. Calculate one of the $SU(3)$ structure constants, say $f_{123}$ using eq. (4.9) in the notes. How would the structure constants change if the generators were renormalized to be a factor 2 larger?

Solution 1. Equation (4.9) states
\[
\begin{bmatrix}
\frac{\lambda_a}{2}, \frac{\lambda_b}{2}
\end{bmatrix} = i f_{abc} \frac{\lambda_b}{2}.
\]

For $a = 1, b = 2$ we have
\[
\frac{1}{2} \begin{pmatrix}
\sigma_1 & 0 \\
0 & 0
\end{pmatrix} \frac{1}{2} \begin{pmatrix}
\sigma_2 & 0 \\
0 & 0
\end{pmatrix} - \frac{1}{2} \begin{pmatrix}
\sigma_2 & 0 \\
0 & 0
\end{pmatrix} \frac{1}{2} \begin{pmatrix}
\sigma_1 & 0 \\
0 & 0
\end{pmatrix} = \frac{1}{4} 2i \epsilon_{123} \begin{pmatrix}
\sigma_3 & 0 \\
0 & 0
\end{pmatrix} = i \frac{1}{2} \lambda_3,
\]
consistent with $f_{123} = 1$.

If all generators would be a factor 2 larger, the commutator (left hand side) would be a factor 4 larger, and the right hand side a factor 2 larger. To maintain equality the structure constants would have to be multiplied by 2. $f_{abc} \rightarrow 2 f_{abc}$. The structure constants are thus not completely constant for a given Lie group, since for example $2 \times$(the generators) still generate the same group. They are however invariant under group rotations.

Problem 2. Convince yourself that the $SU(3)$ step operators in eq. (4.12) with relative angle $60^\circ$ commute, whereas the operators with angle $120^\circ$ do not.

Solution 2. Consider for example
\[
[T_+, V_+]
\]
using $T_\pm = T_1 \pm iT_2$ and $V_\pm = V_1 \pm iV_2$ and the explicit form of the matrices $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ we find
\[
T_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]
Similarly, \( V_+ \) which has \( \sigma_i \) matrices spread over the 1 and 3 rows/columns is obtained by swapping the 1, 2 rows/columns from the \( T_+ \) case, to the 1, 3 rows/columns thus

\[
V_+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\] (5)

and for \( U_+ \) (with \( \sigma_i \) matrices in the 2 and 3 rows/columns)

\[
U_+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}
\] (6)

Similarly, or by using \((T_\pm)^\dagger = T_\mp\), etc., we have for \( T_-, V_, U_- \)

\[
T_- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad U_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad V_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\] (7)

Inspecting the matrices, we see that multiplying \( U_+, V_+ \) in any order, we must get 0. Similarly \( T_+, V_+ \), and \( U_-, T_+ \) give 0, when multiplied in any order. Thus

\[
[U_+, V_+] = [T_+, V_+] = [U_-, T_+] = 0.
\] (8)

Taking the hermitian conjugate of this we get

\[
[U_-, V_-] = [T_-, V_-] = [U_+, T_-] = 0.
\] (9)

On the other hand for \([U_-, V_+]\) we find

\[
\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \gamma_3 - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = T_+.
\] (10)

Similarly \([T_+, V_-] = -U_m, [T_+, U_+] = V_+\).

**Problem 3.** Prove that the totally antisymmetric tensor \( \epsilon^{ijkl} \) is an invariant in special relativity. Use that we are considering a special group, \( SO(1,3)^\dagger \), with determinant one, and the Leibniz formula for the determinant,

\[
\det(A) = \sum_{\sigma \in S_n} \text{sign}(\sigma) \prod_{i=1}^{n} A_{i\sigma(i)}.
\] (11)

For \( SU(3) \) there is a similar condition for a tensor with three indices. What does that have to do with hadron physics?
Solution 3. Letting \( \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \) denote a permutation of \( n \) indices \( 1 \ldots n \), we have for the determinant of an \( n \times n \) matrix \( A \) (the Leibniz formula)

\[
\det(A) = \sum_{\sigma \in S_n} \text{sign}(\sigma) \prod_{i=1}^{n} A_{i \sigma(i)} .
\]

(12)

Writing the sign and the sum in terms of \( \epsilon_{i_1 \ldots i_n} \) we have (with the Einstein summation convention)

\[
\det(A) = \sum_{i_1, \ldots, i_n} \epsilon_{i_1 \ldots i_n} A_{1i_1} A_{2i_2} \ldots A_{ni_n} = A_{1i_1} A_{2i_2} \ldots A_{ni_n} \epsilon_{i_1 \ldots i_n} ,
\]

(13)

Viewing the rank 4, totally anti-symmetric tensor in special relativity, as a tensor, and transforming it as a tensor (i.e., with one \( \Lambda \) for each index), we get

\[
\epsilon_{ijkl} \rightarrow \Lambda^i_{i'} \Lambda^j_{j'} \Lambda^k_{k'} \Lambda^l_{l'} \epsilon_{i'j'k'l'} .
\]

(14)

In particular for \( \epsilon_{1234} \) we have

\[
\epsilon_{1234} \rightarrow \Lambda^1_{i'} \Lambda^2_{j'} \Lambda^3_{k'} \Lambda^4_{l'} \epsilon_{i'j'k'l'} .
\]

(15)

This is just a special case of the determinant equation, eq. (13), for \( n = 4 \). Since the determinant is invariant, \( \epsilon_{1234} \) must be as well. Other components, such as \( \epsilon_{2134} \) are trivially obtained by reordering. Thus \( \epsilon_{ijkl} \) must be an invariant as a consequence of the constant determinant condition.

In most group theory books and courses, matrix groups are characterized by their matrix form. An alternative way of classifying groups, is to describe them in terms of what objects they leave invariant, for \( SO(1,3) \) these objects are the Levi-Cevita tensor with four indices, and the metric.

For QCD, the relevant group is \( SU(3) \) which of course leave \( \epsilon_{ijk} \) invariant. The colorless three quarks state in QCD is often written

\[
\frac{1}{\sqrt{6}} (|rgb \rangle + |brg \rangle + |gbr \rangle - |rbg \rangle - |bgr \rangle - |grb \rangle) ,
\]

(16)

using position to distinguish the different quarks. Letting, for example \( r = 1, g = 2, b = 3 \), this can alternatively be written

\[
\frac{1}{\sqrt{6}} (\epsilon^{123} |q_1q_2q_3 \rangle + \epsilon^{312} |q_3q_1q_2 \rangle + \epsilon^{231} |q_2q_3q_1 \rangle + \epsilon^{132} |q_1q_3q_2 \rangle + \epsilon^{321} |q_3q_2q_1 \rangle + \epsilon^{213} |q_2q_1q_3 \rangle) ,
\]

(17)

or in terms of the totally antisymmetric tensor

\[
\frac{1}{\sqrt{6}} \epsilon^{ijk} |q_iq_jq_k \rangle .
\]

To prove the invariance of this state, we will use the above form and make a rotation. Since \( \epsilon^{ijk} \) is invariant, we let it stand, and just rotate the color state with an \( SU(3) \) group element \( g \):
\[ \epsilon^{ijk} |q_iq_jq_k \rangle \rightarrow \epsilon^{ijk} g_{ii'}g_{jj'}g_{kk'} |q_i'q_j'q_k' \rangle \]  
(18)

(To convince yourself that this is the right thing to do, think about one component at the time, for example \(|q_1 \rangle \rightarrow g_{1i'} |q_i' \rangle \), and think of \(\epsilon^{ijk}\) as just giving a constant, \(\pm 1\), as in eq. (17)). Now, taking the \(g_{ii'}g_{jj'}g_{kk'}\) to act backwards on the \(\epsilon^{ijk}\) instead, results in a transformed \(\epsilon^{ijk}\) (equal to itself). Therefore

\[ \epsilon^{ijk} |q_iq_jq_k \rangle \rightarrow \epsilon^{ijk} g_{ii'}g_{jj'}g_{kk'} |q_i'q_j'q_k' \rangle = \epsilon^{ij'k'} |q_i'q_j'q_k' \rangle, \]  
(19)

where we have used that

\[ (\epsilon^{ijk} g_{ii'}g_{jj'}g_{kk'})^* = \epsilon^{ijk} (g_{ii'}g_{jj'}g_{kk'})^* = \epsilon^{ij'k'} g_{i'i}g_{j'j}g_{k'k} \]  
(20)

is invariant.

**Problem 4.** The Dirac spinor is the object that transforms as a \((0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)\) representation of the Lorentz group. Find the transformation of such an object under:

(i) Parity

(ii) Charge conjugation

**Solution 4.** (i) Parity.

Under the parity transformation, (see notes) the generators \(N_i^\pm = \frac{L_i \pm i K_i'}{2}\) transform into each other, i.e.

\[
\text{Parity: } N_i^+ \leftrightarrow N_i^- 
\]

since \(L_i \rightarrow L_i\) and \(K_i' \rightarrow -K_i'\). Therefore, the \((0, \frac{1}{2})\) representation of a transformation, becomes the \((\frac{1}{2}, 0)\) representation of this transformation and vice versa under parity transformations.

Rotational transformations look the same for both representations, but boost transformations differ by a sign and it is easy to make the above statement explicit

\[
\left( \Lambda_{\vec{K}'} \right)_{\frac{1}{2}, 0} = e^{\vec{\phi} \cdot \vec{K}'} \rightarrow \underset{p}{e^{-\vec{\phi} \cdot \vec{K}'} = \left( \Lambda_{\vec{K}'} \right)_{0, \frac{1}{2}}} 
\]

\[
\left( \Lambda_{\vec{K}'} \right)_{0, \frac{1}{2}} = e^{-\vec{\phi} \cdot \vec{K}'} \rightarrow \underset{p}{e^{\vec{\phi} \cdot \vec{K}'} = \left( \Lambda_{\vec{K}'} \right)_{\frac{1}{2}, 0}} 
\]

Therefore, if we want to describe a physical system that is invariant under parity transformations, we will always need right-chiral and left-chiral spinors. The easiest thing to do is to write them below each other into a single object called Dirac spinor

\[
\Psi = \begin{pmatrix} \xi_R \\ \chi_L \end{pmatrix} = \begin{pmatrix} \xi^a \\ \chi_a \end{pmatrix}
\]

\(^1\)Different conventions exists for what spinor (left or right) is written upstairs.
Without imposing the Dirac equation or and left-right symmetry, there is no connection between \( \chi \) and \( \xi \). A Dirac spinor of the form

\[
\Psi_M = \begin{pmatrix} \chi_R \\ \chi_L \end{pmatrix}
\]
i.e., having the right-chiral spinor obtained directly from the left-chiral spinor, \( \chi_R = \chi_L^C \), in the upper component is a special case, called a Majorana spinor, popularly expressed as a particle being its own anti-particle. A Dirac or Majorana spinor is not a four-vector, because it transforms completely different. A Dirac spinor transforms according to the \( (0, \frac{1}{2}) \oplus (\frac{1}{2}, 0) \) representation of the Lorentz group, which means nothing more than writing the corresponding transformations in block-diagonal form into one big matrix

\[
\Psi \rightarrow \Psi' = \Lambda_{(0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)} \Psi = \begin{pmatrix} \Lambda_{(0, \frac{1}{2})} & 0 \\ 0 & \Lambda_{(\frac{1}{2}, 0)} \end{pmatrix} \begin{pmatrix} \xi_R \\ \chi_L \end{pmatrix}
\]

Let us now see how a Dirac spinor transforms under parity. If a Dirac spinor transforms under \( (0, \frac{1}{2}) \oplus (\frac{1}{2}, 0) \), the parity transformed object transforms according to the \( (\frac{1}{2}, 0) \oplus (0, \frac{1}{2}) \) representation. Therefore,

\[
\Psi = \begin{pmatrix} \xi_R \\ \chi_L \end{pmatrix} \rightarrow \Psi^P = \begin{pmatrix} \chi_L \\ \xi_R \end{pmatrix}
\]

A parity transformed Dirac spinor contains the same objects as the untransformed Dirac spinor, only written differently.

Note that this is also consistent with the parity operator being \( \gamma^0 \) (which is something you may know from a course in particle physics), which in the chiral (or Weyl) representation is \( \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \) (possibly up to a phase, depending on convention).

(ii) Charge Conjugation

Charge conjugation is the transformation that yields \( \chi_L \rightarrow \chi_R \) and \( \xi_R \rightarrow \xi_L \), i.e. \( \chi_L \rightarrow \chi_L^C = \epsilon \chi_L^* = \chi_R \) and analogously for the right-chiral spinor. How does a Dirac spinor transform under such a transformation? Naively we would get

\[
\Psi = \begin{pmatrix} \xi_R \\ \chi_L \end{pmatrix} \rightarrow \bar{\Psi} = \begin{pmatrix} \xi_L^C \\ \chi_L^* \end{pmatrix} = \begin{pmatrix} \xi_L \\ \chi_R \end{pmatrix}
\]

Unfortunately, this object does not transforms like a Dirac spinor, since under boosts \( \Psi \rightarrow \Psi' = \begin{pmatrix} e^{-\gamma \cdot \vec{a}} & 0 \\ 0 & e^\gamma \cdot \vec{a} \end{pmatrix} \Psi \) and \( \bar{\Psi} \rightarrow \bar{\Psi}' = \begin{pmatrix} e^{\bar{\gamma} \cdot \vec{a}} & 0 \\ 0 & e^{-\bar{\gamma} \cdot \vec{a}} \end{pmatrix} \bar{\Psi} \)

(Contrary to the parity transformation, charge conjugation is not a spacetime symmetry, and therefore our Lorentz transformations are unchanged.) This is a different kind
of object, because it transforms according to a different representation of the Lorentz group. Therefore, we write

\[ \Psi = \begin{pmatrix} \xi_R \\ \chi_L \end{pmatrix} \rightarrow \Psi_C = \begin{pmatrix} \chi^C_L \\ \xi^C_R \end{pmatrix} = \begin{pmatrix} \chi_R \\ \xi_L \end{pmatrix} \]

which incorporates the transformation behavior we observed earlier and transforms like a Dirac spinor. This operation is called charge conjugation, which can be a little misleading, since we do more than just charge conjugating the Weyl spinors. We know that this transforms a left-chiral spinor into a right-chiral, i.e. flips one label we use to describe our elementary particles. This operator flips not only one, but all labels we use to describe fundamental particles. One such label is electric charge, hence the name charge conjugation.

**Problem 5.** Show that \( SL(2; \mathbb{C}) \) is a double cover of the proper orthochronous Lorentz group.

Associate with each four vector \( x^\mu \) a hermitian matrix

\[ x_\alpha \sigma^\alpha = \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix}. \]

**Hint:** Exercise sheet 1, problem 4.

**Solution 5.** In what follows we establish the fact that elements of the proper, orthochronous Lorentz group \( SO(1,3)^\uparrow \) can be described by means of elements of \( SL(2; \mathbb{C}) \), the group of all \( 2 \times 2 \) complex matrices

\[ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ with } \det g = ad - bc = 1. \]

In the natural topology of matrices the group \( SL(2; \mathbb{C}) \) is simply connected. The relation between the two groups can be established as follows. One associates with each four vector \( x^\mu \) a hermitian matrix

\[ Q = x_\alpha \sigma^\alpha = \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix} \]

with \( \sigma^0 = \mathbb{I}_2 \) and \( \sigma^i \ (i = 1, 2, 3) \) the Pauli matrices. In this way one defines a one-to-one linear correspondence between all four-vectors and all \( 2 \times 2 \) hermitian matrices. (This is very similar to what we did in the \( SU(2) \) case.) We are essentially considering the isomorphism

\[ f : \mathbb{R}^4 \rightarrow H_2 \]

\[ f(\hat{e}_i) = \sigma_i. \]

It is often also very convenient to parametrize the elements \( g \) of the group \( SL(2; \mathbb{C}) \) by

\[ g = g_\mu \sigma^\mu, \]

with \( g_\mu \) being complex numbers.
Corresponding to every element $g$ of the group $\text{SL}(2; \mathbb{C})$ consider the following transformation in the space of the hermitian matrices $Q$:

$$Q' = gQg^\dagger$$

with $Q'_\alpha = x'_\alpha \sigma^\alpha$. The corresponding operation in the Minkowsky space of four-vectors is a linear transformation

$$x'^\alpha = \Lambda^\alpha_\beta (g) x^\beta \quad (x' = \Lambda(g)x)$$

where the transformation matrix $\Lambda$ can be expressed in terms of the matrix $g$ of the group $\text{SL}(2; \mathbb{C})$. These transformations preserves the scalar product since

$$x'^\mu x'^\mu = (x'^0)^2 - \sum_i (x'^i)^2 = \det Q' = \det Q = (x^0)^2 - \sum_i (x^i)^2 = x^\mu x_\mu .$$

The matrix elements $\Lambda^\alpha_\beta$ can be expressed in terms of the corresponding matrix $g$ of the group $\text{SL}(2; \mathbb{C})$. Using the properties of the Pauli matrices one has

$$\Lambda x = f^{-1}(g f(x) g^\dagger)$$

$$\Leftrightarrow f(\Lambda x) = g f(x) g^\dagger$$

$$\Leftrightarrow (\Lambda x)^\alpha \sigma_\alpha = g x^\beta \sigma_\beta g^\dagger$$

$$\Leftrightarrow \Lambda^\alpha_\beta x^\beta \sigma_\alpha = g x^\beta \sigma_\beta g^\dagger$$

$$\Rightarrow \Lambda^\alpha_\beta \sigma_\alpha = g \sigma_\beta g^\dagger$$

$$\Leftrightarrow \Lambda^\alpha_\beta = \frac{1}{2} \text{Tr} (\sigma_\alpha g \sigma_\beta g^\dagger) .$$

The above equation shows that to an arbitrary matrix $g$ of $\text{SL}(2; \mathbb{C})$ there corresponds a $4 \times 4$ matrix $\Lambda$.

Note that the $g = g_\mu \sigma^\mu$ and $-g$ are mapped to the same $\Lambda^\alpha_\beta$, hence, if we have a cover of $\text{SO}(1, 3)^\dagger$, it is a double cover.

The explicit expression of the transformation $\Lambda(g)$ in terms of the parameters $g_0$ and $g_k$ of the matrix $g$ is as follows

$$\Lambda_0^0 = |g_0|^2 + \sum_{k=1}^3 |g_k|^2$$

$$\Lambda_0^k = g_0 g_k^* + g_k^* g_0 - i \epsilon^{klm} g_l h_m^*$$

$$\Lambda_0^k = g_0 g_k^* + g_k^* g_0 + i \epsilon^{klm} g_l h_m^*$$

$$\Lambda_0^l = \delta_k^l \left( |g_0|^2 - \sum_{s=1}^3 |g_s|^2 \right) + g_k g_l^* + g_l^* g_k - i \epsilon^{klm} (g_0 g_m^* - g_0 g_m^*)$$

We now show that the matrix $\Lambda$ belongs to the proper orthochronous Lorentz group $\text{SO}(1, 3)^\dagger$. First, we saw that the quadratic form $x^\mu x_\mu$ is invariant under the transformation $\Lambda$, and therefore the matrix $\Lambda$ is an element of the $O(1, 3)$ Lorentz group. Noting that $\Lambda_0^0 \geq 1$ we conclude that we get an element in $O(1, 3)^\dagger$. 7
It remains to argue that the determinant is one. For the special case for which $g$ is the $2 \times 2$ unit matrix, the corresponding $\Lambda$ is the identity transformation, and hence, in this case, $\det \Lambda = 1$. Since $\det(\Lambda)$ is a continuous function of the four variables of the matrix $g$ of the group $\text{SL}(2, C)$, and since the domain of variation of these four variables is simply connected, a discontinuous jump from $\det \Lambda = 1$ to $\det \Lambda = -1$ is excluded. Consequently, $\det \Lambda = 1$ for all indices. Hence $\Lambda$ belongs to the proper Lorentz group. Consequently, $\Lambda$ is an element of the proper, orthochronous, Lorentz group $\text{SO}(1, 3)^\dagger$.

One also notices that because the group $\text{SL}(2; C)$ is connected, and the mapping into the proper orthochronous Lorentz group is a continuous homomorphism, the image of the group $\text{SL}(2; C)$ must be a subgroup of the proper orthochronous, Lorentz group $\text{SO}(1, 3)^\dagger$. Noting that we do get both rotations and boosts, we conclude that the whole group $\text{SO}(1, 3)^\dagger$ must be reached.