The Hamilton formalism

So far we have encountered the Newtonian formulation of mechanics and the Lagrangian. Now we will study yet another formalism, the Hamilton formalism, where

\[ H(\bar{q}, \bar{p}) = \sum p_i \dot{q}_i (\bar{q}, \bar{p}) - L \]

is the basic object. Note that we view H as depending on the (generalized coordinates \( \bar{q} \) and) generalized momenta \( \bar{p} \), rather than generalized velocities \( \dot{q}_i \).

The 2N-dimensional space given by \( \{\bar{q}, \bar{p}\} \) we call the phase space.

For simplicity, we start with the case of one generalized coordinate

\[ L = L(\bar{q}, \dot{\bar{q}}, t) = T(\bar{q}, \dot{\bar{q}}, t) - V(\bar{q}, t) \]

In natural coordinates our Lagrangian has the form

\[ L \approx \frac{1}{2} A_{ij}(\bar{q}) \dot{q}_i \dot{q}_j - V(\bar{q}) \]

From the definition of \( p \) we have

\[ p_i = \frac{\partial L}{\partial \dot{q}_i} = A_{ij}(\bar{q}) \dot{q}_j \]

meaning that the Hamiltonian is

\[ H = \sum p_i \dot{q}_i - L = \sum A_{ij}(\bar{q}) \dot{q}_i \dot{q}_j - \left( \sum \frac{1}{2} A_{ij}(\bar{q}) \dot{q}_i \dot{q}_j - V(\bar{q}) \right) \]

\[ = 2T - (T - V) = T + V = E \]

i.e. for natural systems the Hamiltonian is just the energy.
To view $H$ as depending on $\dot{p}$ rather than $\dot{q}$ we solve for $\dot{q}$

$$\dot{q} = \frac{\partial}{\partial q} = \dot{q}(q, p)$$

$$H(q, p) = \sum_i p_i \dot{q}_i(q, p, t) - L(q, \dot{q})(q, p, t)$$

From this Hamiltonian we will derive a new set of equations describing the motion, Hamilton's equations.

First we differentiate $H$ w.r.t. $q$:

$$\frac{\partial H}{\partial q_i} = \sum_j p_j \frac{\partial \dot{q}_j}{\partial q_i} - \left[ \sum_j \frac{\partial L}{\partial q_i} + \sum_j \frac{\partial L}{\partial \dot{q}_i} \dot{q}_j \right]$$

$$= -\frac{\partial L}{\partial q_i} = -\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) = -\dot{p}_i$$

Next, differentiate w.r.t. $p$ instead

$$\frac{\partial H}{\partial p_i} = \dot{q}_i + \sum_j p_j \frac{\partial \dot{q}_j}{\partial p_i} - \sum_j \frac{\partial L}{\partial q_i} \dot{q}_i$$

In total we have Hamilton's equations

$$\frac{\partial H}{\partial q_i} = -\dot{p}_i \quad \frac{\partial H}{\partial p_i} = \dot{q}_i$$

Note: In the Lagrangian approach we had one 2nd order equation.

In the Hamiltonian approach we have two 1st order equations.
Hamilton's equations in several variables

Let's assume that we have as many coordinates as the system has d.o.f. forces can be derived from the potential energy.

(We may let the coordinate transformation from Cartesian to generalized coordinates depend on time.)

Consider the Lagrangian

\[ L = L(q, \dot{q}, t) = T - V \]

where

\[
\begin{cases}
\bar{q} = (q_1, \ldots, q_n) \\
\bar{\dot{q}} = (\dot{q}_1, \ldots, \dot{q}_n)
\end{cases}
\]

We have the Euler-Lagrange equations

\[ \frac{\partial L}{\partial q_i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) = 0, \quad i = 1 \ldots n \]

and the Hamiltonian is

\[ H = \sum_{i=1}^{n} p_i \dot{q}_i - L \]

where

\[ p_i = \frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}_i} \quad i = 1 \ldots n \]

Now we want to express our Hamiltonian in the 2n variables \( \bar{q}, \bar{p} \), thus we rewrite:

\[ \dot{q}_i = \dot{q}_i(q_1, \ldots, q_n, p_1, \ldots, p_n, t) \quad i = 1 \ldots n \]

or:

\[ \dot{\bar{q}} = \dot{\bar{q}}(\bar{q}, \bar{p}, t) \]

giving us the Hamiltonian

\[ H = H(\bar{q}, \bar{p}, t) = \sum_{i=1}^{n} p_i \dot{q}_i(q_1, \ldots, q_n, p_1, \ldots, p_n, t) - L(q_1, \ldots, q_n, p_1, \ldots, p_n, t) \]
Consider the time derivative of $H$. 

Naively $H$ can vary with time because

- $H$ changes as the coordinates change with time
- $H$ changes if it depends explicitly on time

$s.t.$

$$\frac{dH}{dt} = \sum_i \left( \frac{\partial H}{\partial q_i} \dot{q}_i + \frac{\partial H}{\partial p_i} \dot{p}_i \right) + \frac{\partial H}{\partial t}$$

However, due to Hamilton’s equations the first term vanishes for each $i$

$$\left( \frac{\partial H}{\partial q_i} \dot{q}_i + \frac{\partial H}{\partial p_i} \dot{p}_i \right) = \frac{\partial H}{\partial q_i} \frac{\partial H}{\partial q_i} + \frac{\partial H}{\partial p_i} \frac{\partial H}{\partial p_i} \left( -\frac{\partial H}{\partial q_i} \right) = 0 \quad \forall i$$

i.e. $H$ only changes with time if it depends explicitly on time.

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}$$

In particular, if $H$ does not depend explicitly on time $H$ is conserved.

Recall that for natural coordinates $H=E$, so energy is conserved if

(However, the relation between the generalized coordinates and Cartesian coordinates does not involve time)

H does not depend explicitly on time.
Example: The Atwood machine

For the Lagrangian we have

\[ L = T - U = \frac{1}{2} (m_1+m_2) \dot{x}^2 - (m_1-m_2)gx \]

\[ H \equiv p \dot{x} - L = \left( \frac{2L}{\partial \dot{x}} \right) \dot{x} - L \\
= (m_1+m_2) \dot{x}^2 - \frac{1}{2} (m_1+m_2) \dot{x}^2 - (m_1-m_2)gx \\
= \frac{1}{2} (m_1+m_2) \dot{x}^2 - (m_1-m_2)gx \]

Indeed \( H = E \), as it should be for natural systems.

Alternatively, knowing that we have natural generalized coordinates, we could have written down

\[ H = E = T + U = \frac{1}{2} (m_1+m_2) \dot{x}^2 - (m_1-m_2)gx \]

To use Hamilton’s equations, we first calculate the generalized momentum

\[ p = \frac{2L}{\partial \dot{x}} = \frac{2T}{\partial \dot{x}} = (m_1+m_2) \dot{x} \]

To write down \( H \) in terms of \( q \) and \( p \), we solve for \( \dot{x} = \frac{p}{m_1+m_2} \)

\[ \Rightarrow H(q,p) = \frac{1}{2} (m_1+m_2) \left( \frac{p}{m_1+m_2} \right)^2 - (m_1-m_2)gx \]

\[ = \frac{1}{2} \frac{p^2}{m_1+m_2} - (m_1-m_2)gx \]

Giving the Hamilton equations

\[ \dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m_1+m_2} \]

\[ \dot{p} = -\frac{\partial H}{\partial x} = -(m_1-m_2)g \]

\[ \ddot{x} = \frac{p}{m_1+m_2} = \frac{(m_1-m_2)g}{m_1+m_2} \] as before
Hamilton's equations for a particle in a central force field

Find Hamilton's equations for a particle of mass \( m \) moving in a central force field \( V(r) \).

The kinetic energy is

\[
T = \frac{1}{2} m \left( \dot{r}^2 + r^2 \dot{\phi}^2 \right)
\]

giving

\[
\begin{align*}
\dot{p}_r &= \frac{\partial L}{\partial \dot{r}} = m \dot{r} \\
\dot{p}_\phi &= \frac{\partial L}{\partial \dot{\phi}} = m r^2 \dot{\phi}
\end{align*}
\]

\( m \) (radial velocity) \hspace{1cm} \text{(angular momentum)}

To express \( H \) in terms of \( \mathbf{\hat{q}}, \mathbf{\hat{p}} \) we solve for \( \dot{r}, \dot{\phi} \)

\[
\begin{align*}
\dot{r} &= \frac{\dot{p}_r}{m} \\
\dot{\phi} &= \frac{\dot{p}_\phi}{m r^2}
\end{align*}
\]

\[ H = T + V = \frac{1}{2} m \left[ \left( \frac{\dot{p}_r}{m} \right)^2 + r \left( \frac{\dot{p}_\phi}{m r^2} \right)^2 \right] + V(r) \]

\[ = \frac{1}{2m} \left[ p_r^2 + \frac{p_\phi^2}{r^2} \right] + V(r) \]

From \( H \) we get 2*2 1st order Hamilton equations:

\[
\begin{align*}
\dot{p}_r &= \frac{\partial H}{\partial \dot{r}} = \frac{p_r}{m} \\
\dot{p}_\phi &= \frac{\partial H}{\partial \dot{\phi}} = -\frac{2mr}{m} = 0 \tag{angular momentum \( p_\phi \) is conserved}
\end{align*}
\]

\[ \dot{r} = \frac{2H}{\partial p_r} = \frac{p_r}{m} \]

\[ \dot{\phi} = \frac{\partial H}{\partial \dot{\phi}} = \frac{p_\phi}{m r^2} \]

\[ \begin{align*}
\dot{r} &= \frac{2H}{\partial p_r} = \frac{p_r}{m} & \text{(reproduces def. of radial momentum)} \\
\dot{\phi} &= \frac{\partial H}{\partial \dot{\phi}} = \frac{p_\phi}{m r^2} & \text{(most interesting equation)}
\end{align*} \]

If we substitute the derivative of \( \frac{\partial H}{\partial \dot{r}} \) into \( \frac{\partial H}{\partial \dot{\phi}} \) we get

\[
\dot{r} = \frac{\partial H}{\partial \dot{r}} \Rightarrow m \ddot{r} = \frac{p_r^2}{m r^3} - \frac{2V}{r}
\]

\( \text{centrifugal force} \hspace{1cm} \text{actual force} \)
In general the procedure for setting up Hamilton's equations are:

1) Choose generalized coordinates

2) Write down T and V in terms of $q_i, \dot{q}_i$ $V$ independent of $\dot{q}_i$

3) Find the generalized momenta

4) Solve for $q_i$ in terms of $q_i, p_i$

5) Write down the Hamiltonian in terms of $q_i, p_i$

   If the coordinates are natural $H = T + V$.

6) Write down and solve Hamilton's equations
Cyclic (ignorable) coordinates

We have seen for the Lagrangian that if $L$ is independent of $q_i$ (i.e. $q_i$ is a cyclic or ignorable coordinate)
the corresponding conjugated momentum $p_i = \frac{\partial L}{\partial \dot{q}_i}$ is conserved.
In the same way it follows from Hamiltons equations $\dot{p}_i = -\frac{\partial H}{\partial q_i}$
that if $H$ is independent of $q_i$ the corresponding $p_i$ is conserved.

This is actually the same result since
$$\frac{\partial H}{\partial q_i} = \sum_{k=1}^n \frac{\partial x_i}{\partial q_j} \frac{\partial p_j}{\partial q_i} - \frac{\partial L}{\partial q_i} - \sum_{k=1}^n \frac{\partial x_j}{\partial q_i} \frac{\partial p_j}{\partial q_i} = -\frac{\partial L}{\partial q_i}$$

meaning that $H$ is independent of $q_i$ iff $L$ is independent of $q_i$.

For cyclic coordinates the Hamiltonian formalism has an advantage over the Lagrangian:

Let $H = H(q_1, q_2, p_1, p_2)$ where $q_2$ is a cyclic coordinate, s.t.:
$$H = H(q_1, p_1, p_2)$$
but since $q_2$ is cyclic $\dot{q}_2 = -\frac{\partial H}{\partial q_2} = 0 \Rightarrow p_2 = \text{const}$
$$\Rightarrow H = H(q_1, p_1, \text{const}) = H(q_1, p_1)$$
so the system has reduced to a one-dimensional system.

Note that this is not true in the Lagrangian formalism where
$$L = L(q_1, q_2, \dot{q}_1, \dot{q}_2) = L(q_1, \dot{q}_1, \dot{q}_2)$$
since $\dot{q}_2$ is not necessarily constant.