Not quite in Rindler, partly covered in HUB, p. 157

A general Lorentz boost

Consider a boost in a general direction:

simply $\beta^1 x^1$ for boost in x-direction

take $\bar x$ component in $\bar\beta$ dir.

The time component must change as $x'^0 = \gamma(x^0 - \bar\beta\cdot\bar x)$ for $\bar\beta = \frac{\bar v}{c}$

The components orthogonal to the direction of motion don't change

$$\bar x'_\perp = \bar x_\perp$$

and the components parallel to the direction of motion change as

$$\bar x'_\parallel = \gamma(\bar x_\parallel - \bar\beta tc)$$ as $\bar\beta$ is in the same direction as $\bar x_\parallel$

We can rewrite $\bar x'$ as

express in $\bar x$

$$\bar x' = \bar x'_\perp + \bar x'_\parallel = \underbrace{\bar x_\perp}_{\bar x - \bar x_\parallel} + \gamma(\bar x_\parallel - \bar\beta \underbrace{tc}_{x^0})$$

$$= \bar x - \bar x_\parallel + \gamma(\bar x_\parallel - \bar\beta x^0)$$

collect in front of $\bar x_\parallel$

$$= \bar x + (\gamma - 1)\bar x_\parallel - \gamma\bar\beta x^0$$

$\bar x_\parallel = \frac{\bar\beta\cdot\bar x\,\bar\beta}{\bar\beta^2}$

$$= \bar x + (\gamma-1)\frac{(\bar\beta\cdot\bar x)}{\bar\beta^2}\bar\beta - \gamma\bar\beta x^0$$

$$(x')^1 = x^1 + (\gamma-1)\frac{\overbrace{\bar\beta\cdot\bar x}^{\beta^1x^1+\beta^2x^2+\beta^3x^3}}{\bar\beta^2}\beta^1 - \gamma\beta^1 x^0$$

We may now collect the results into one transformation matrix:

$$\boxed{\begin{array}{c} x'^0 \\ \\ \\ \\ \end{array}\begin{bmatrix} ct' \\ x'^1 \\ x'^2 \\ x'^3 \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta^1 & -\gamma\beta^2 & -\gamma\beta^3 \\ -\gamma\beta^1 & 1+(\gamma-1)\frac{(\beta^1)^2}{\bar\beta^2} & (\gamma-1)\frac{\beta^1\beta^2}{\bar\beta^2} & (\gamma-1)\frac{\beta^1\beta^3}{\bar\beta^2} \\ -\gamma\beta^2 & (\gamma-1)\frac{\beta^1\beta^2}{\bar\beta^2} & 1+(\gamma-1)\frac{(\beta^2)^2}{\bar\beta^2} & (\gamma-1)\frac{\beta^2\beta^3}{\bar\beta^2} \\ -\gamma\beta^3 & (\gamma-1)\frac{\beta^1\beta^3}{\bar\beta^2} & (\gamma-1)\frac{\beta^2\beta^3}{\bar\beta^2} & 1+(\gamma-1)\frac{(\beta^3)^2}{\bar\beta^2} \end{bmatrix} \begin{bmatrix} ct \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}\begin{array}{c} x^0 \\ \\ \\ \\ \end{array}}$$

But this is true for any contravatiant 4-vector => it is true for the 4-momentum!

$$\begin{array}{c} p'^0 \\ \\ \\ \\ \end{array}\begin{bmatrix} E'/c \\ p'_x \\ p'_y \\ p'_z \end{bmatrix} = \begin{bmatrix} \\ & \text{same matrix} & \\ \\ \end{bmatrix} \begin{bmatrix} E/c \\ p_x \\ p_y \\ p_z \end{bmatrix}\begin{array}{c} p^0 \\ \\ \\ \\ \end{array}$$

Invariance of the metric

(Not in Rindler)

Let $x = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$ denote a 4-vector

We know that $(x)^2 = x \cdot x$ is invariat

index form: | matrix form:

$$(x)^2 = \sum_{\mu\nu} x^\mu g_{\mu\nu} x^\nu \Leftrightarrow x^T g x \Leftrightarrow (x^0, x^1, x^2, x^3)\begin{pmatrix} 1 &&& \\ & -1 && \\ && -1 & \\ &&& -1 \end{pmatrix}\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$\parallel \qquad\qquad \parallel$$

$$x'^\mu g'_{\mu\nu} x'^\nu \Leftrightarrow x'^T g' x'$$

$$\parallel \qquad\qquad \parallel$$

$$(\Lambda x)^\mu g'_{\mu\nu} (\Lambda x)^\nu \Leftrightarrow (\Lambda x)^T g' \Lambda x$$

$$x^T \Lambda^T g' \Lambda x$$

$$\Rightarrow \quad g = \Lambda^T g' \Lambda \qquad \left(\text{or } \; g' = (\Lambda^T)^{-1} g \Lambda^{-1}\right)$$

As no system is special we may assume we know $g' = \begin{bmatrix} 1 &&& \\ & -1 && \\ && -1 & \\ &&& -1 \end{bmatrix}$ and check that g is the same (rather than opposite):

Recall for x-boost: $\Lambda = \begin{bmatrix} \gamma & -\gamma \overset{= u/c}{\beta} && \\ -\gamma\beta & \gamma && \\ && 1 & \\ &&& 1 \end{bmatrix} = \Lambda^T$

giving us:

$$\Lambda^T g' \Lambda = \begin{bmatrix} \gamma & -\gamma\beta && \\ -\gamma\beta & \gamma && \\ && 1 & \\ &&& 1 \end{bmatrix} \overbrace{\begin{bmatrix} 1 &&& \\ & -1 && \\ && -1 & \\ &&& -1 \end{bmatrix}}^{g'} \begin{bmatrix} \gamma & -\gamma\beta && \\ -\gamma\beta & \gamma && \\ && 1 & \\ &&& 1 \end{bmatrix}$$

$$= \begin{bmatrix} \gamma & -\gamma\beta && \\ -\gamma\beta & \gamma && \\ && 1 & \\ &&& 1 \end{bmatrix} \begin{bmatrix} \gamma & -\gamma\beta && \\ \gamma\beta & -\gamma && \\ && -1 & \\ &&& -1 \end{bmatrix}$$

$$= \begin{bmatrix} \overbrace{\gamma^2(1-\beta^2)}^{1} & \cancel{-\gamma\gamma\beta + \gamma\beta\gamma} && \\ \cancel{-\gamma\beta\gamma + \gamma\gamma\beta} & \gamma^2(\beta^2-1) && \\ && -1 & \\ &&& -1 \end{bmatrix} = \begin{bmatrix} 1 &&& \\ & -1 && \\ && -1 & \\ &&& -1 \end{bmatrix} = g$$

OK, g invariant despite transforming as a tensor => g special!

This shouldn't be a surprise, we have already seen that a Lorentz boost leaves $(x)^2$ invariant.

General Lorentz transformations

HUB 146-152
(II.8)

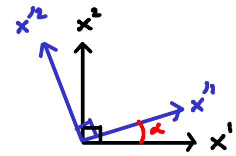
It makes sense to ask what is the most general linear transformation that leaves the metric invariant. In fact it's easy to guess the answer:

1) Lorentz boosts in any direction (Clearly x-direction is not special)

=> 3 degrees of freedom

2) Spatial rotations, we know from linear algebra:

$$\begin{bmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & \cos(\alpha) & \sin(\alpha) & \\ & -\sin(\alpha) & \cos(\alpha) & \\ & & & 1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}$$



(We here use passive rotations, i.e., we rotate the coordinates, not the system.)

$$\Rightarrow (x'^0)^2 - (x'^1)^2 - (x'^2)^2 - (x'^3)^2$$
$$= (x^0)^2 - (\cos(\alpha) x^1 + \sin(\alpha) x^2)^2 - (-\sin(\alpha) x^1 + \cos(\alpha) x^2)^2 - (x^3)^2$$
$$= (x^0)^2 - (x^1)^2 \underbrace{(\cos(\alpha)^2 + \sin(\alpha)^2)} - (x^2)^2 \underbrace{(\cos(\alpha)^2 + \sin(\alpha)^2)} - (x^3)^2$$

and again we may as well rotate in any other plane => 3 degrees of freedom.

3) Space inversion $\bar{x} \rightarrow -\bar{x}$

=> Discrete symmetry => no continuous degree of freedom

4) Time reversal $t \rightarrow -t$

=> Discrete symmetry => no continuous degree of freedom

The set of all transformations above is referred to as the Lorentz transformations, or the Lorentz group. This set of transformations is very important as it leaves the laws of physics invariant. Observers related by Lorentz transformations may disagree on numerical values, but they agree on the form of physical laws.

A group is a well-defined mathematical concept which is very important in theoretical physics, but it's not part of this course. The physically essential properties are that for each transformation there is an inverse transformation in the group and that two transformations after each other also correspond to a transformation in the group. First boosting in x-direction, then in y-direction is the same as first boosting in x-direction (with some larger boost) and then rotating by some angle, so it's not surprising that boosts and rotations form a "group".

Def of group:

Let a,b,c $\in$ G

- $ab \in G \quad \forall\, a,b$
- $\forall a \; \exists a^{-1} \in G$ s.t. $a^{-1}a = 1$
- $(ab)c = a(bc) \quad \forall\, a,b,c$
- there is an identity e, s.t. ea=ae=a $\forall$ a

We search all transformations of form $g = \Lambda^T g \Lambda$

that leave g invariant. (We have proved that this is true for boosts.)

Require this for general Lorentz transformations and see what we get:

$$\Rightarrow \det(\Lambda^T g \Lambda) = \det(g) = -1$$

$$\parallel \quad \det(AB) = \det(A)\det(B) \qquad \text{(no phase)}$$

$$\underbrace{\det(\Lambda^T)}_{\det(\Lambda)} \underbrace{\det(g)}_{-1} \det(\Lambda) \Rightarrow \det(\Lambda^2) = 1 \Rightarrow \det(\Lambda) = \pm 1$$

Transformations with

- $\det(\Lambda) = 1$ are called proper. Rotations and boosts are proper.

- $\det(\Lambda) = -1$ are called improper. They involve space inversion or time reflection.

From now on we will concern ourselves only with proper Lorentz transformations.

=> we are interested in matrices with $\det(\Lambda) = 1$

One can argue that these matrices can be generated by some matrix L with

$Tr(L) = 0$, meaning that

$$\Lambda = e^{L}.$$

def of exp(matrix)

$$\rightarrow \mathbb{1} + L + \frac{L^2}{2!} + \frac{L^3}{3!} + \cdots$$

To see this, assume that L can be diagonalized by some matrix O:

(less important for course)

$$OLO^{-1} = D \qquad \text{for some diagonal matrix } D$$

But then we may rewrite the diagonalized version of Λ:

$$O\Lambda O^{-1} = O e^{L} O^{-1}$$

def of e^M

$$= O\left(\mathbb{1} + L + \frac{L^2}{2!} + \frac{L^3}{3!} + \cdots\right) O^{-1}$$

Insert $\mathbb{1}$s

$$= \mathbb{1} + OLO^{-1} + \frac{1}{2!} OL\overbrace{O^{-1}O}^{\mathbb{1}}LO^{-1} + \frac{1}{3!} OLO^{-1}OLO^{-1}OLO^{-1} + \cdots$$

def of D

$$= \mathbb{1} + D + \frac{1}{2!}D^2 + \frac{1}{3!}D^3 + \cdots$$

$$= \begin{bmatrix} 1+D_{00}+\frac{1}{2}D_{00}^2+\cdots & & & \\ & 1+D_{11}+\frac{1}{2}D_{11}^2+\cdots & & \\ & & 1+D_{22}+\frac{1}{2}D_{22}^2+\cdots & \\ & & & 1+D_{33}+\frac{1}{2}D_{33}^2+\cdots \end{bmatrix}$$

$$= \begin{bmatrix} e^{D_{00}} & & & \\ & e^{D_{11}} & & \\ & & e^{D_{22}} & \\ & & & e^{D_{33}} \end{bmatrix} = \exp(D)$$

Now we have: $\det(\Lambda) = \det(\overbrace{OO^{-1}}^{\mathbb{1}}\Lambda)$

$\det(A)\det B = \det(AB)$

$$= \det(O^{-1}\Lambda O)$$

$$= \det(\exp(D))$$

diagonal matrix -> determinant only product of elements

$$= e^{D_{00}} e^{D_{11}} e^{D_{22}} e^{D_{33}}$$

$$= e^{D_{00}+D_{11}+D_{22}+D_{33}}$$

$$= e^{\mathrm{Tr}[D]}$$

def of D

$$= e^{\mathrm{Tr}[OLO^{-1}]}$$

Trace is cyclic, Tr[A B C]= Tr[C A B]

$$= e^{\mathrm{Tr}[\underbrace{O^{-1}O}_{\mathbb{1}}L]}$$

$$= e^{\mathrm{Tr}[L]} \overset{!}{=} 1$$

So, if we want $\det(\Lambda) = 1$, L has to be traceless!

Using Taylor expansion one can prove

- $\Lambda^T = (e^L)^T = e^{L^T}$ (1)
- $\Lambda^{-1} = (e^L)^{-1} = e^{-L}$ (2)

expand, insert $gg = \mathbb{1}$ and collect again

Using this we may rewrite: $g\Lambda^T g \overset{(1)}{=} g\, e^{L^T} g = e^{gL^T g}$

But we also have: $g\Lambda^T g = g\underbrace{\Lambda^T g \Lambda}_{g}\Lambda^{-1} = g^2\Lambda^{-1} = \Lambda^{-1} \overset{(2)}{=} e^{-L}$

From which we conclude:

right-multiply with g^{-1}

$$gL^T g = -L \Leftrightarrow gL^T = -L\underbrace{g^{-1}}_{g} \Leftrightarrow (Lg)^T = -Lg$$

Lg is an antisymmetric matrix. The most general form of L is then:

(for proper Lorentz transformations)

$$\Rightarrow \quad L = \begin{bmatrix} 0 & L_{01} & L_{02} & L_{03} \\ L_{01} & 0 & L_{12} & L_{13} \\ L_{02} & -L_{12} & 0 & L_{23} \\ L_{03} & -L_{13} & -L_{23} & 0 \end{bmatrix}$$

$$\Rightarrow Lg = \begin{bmatrix} 0 & L_{01} & L_{02} & L_{03} \\ L_{01} & 0 & L_{12} & L_{13} \\ L_{02} & -L_{12} & 0 & L_{23} \\ L_{03} & -L_{13} & -L_{23} & 0 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix} = \begin{bmatrix} 0 & -L_{01} & -L_{02} & -L_{03} \\ L_{01} & 0 & -L_{12} & -L_{13} \\ L_{02} & L_{12} & 0 & -L_{23} \\ L_{03} & L_{13} & L_{23} & 0 \end{bmatrix}$$

Thus in general L can be written using 6 parameters

$$L = +\,\bar{\omega}\cdot\bar{J} - \bar{\eta}\cdot\bar{K}$$

$$\bar{\omega} = (\omega_1, \omega_2, \omega_3), \quad \bar{\eta} = (\eta_1, \eta_2, \eta_3)$$

$$\underset{23}{J_1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad \underset{13}{J_2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad \underset{12}{J_3} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underset{01}{K_1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \underset{02}{K_2} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \underset{03}{K_3} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

(I define L with another sign compared to HUB, to be consistent with the rotation on p. 3.)

(Only in 3 dimensions can a rotation be labeled by one axis, in general labeled by the directions that mix.)

The matrices J_1, J_2, J_3 generate rotations and K_1, K_2, K_3 generate boosts.

To see this consider for example a boost in the x-direction i.e.

$$\left.\begin{aligned}\bar{\eta} &= (\eta, 0, 0)\\ \bar{\omega} &= \bar{0}\end{aligned}\right\} \Rightarrow L = -\eta K_1 = -\eta \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow L^2 = (-\eta)^2 \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \eta^2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For the upper left 2x2 block of Λ we then have $\ell = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$e^{-\eta \ell} = \sum_{n=0}^{\infty} \frac{1}{n!} \ell^n (-\eta)^n \qquad \left[\ell^2 = 1\right]$$

$$= \sum_{n=0}^{\infty} \frac{1}{(2n)!} \underbrace{\ell^{2n} (-\eta)^{2n}}_{\eta^{2n} \mathbb{1}_{2\times 2}} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \underbrace{\ell^{2n+1} (-\eta)^{2n+1}}_{-\eta^{2n+1} \ell}$$

even terms   odd terms

recognize Taylor series of cosh and sinh

$$= \cosh(\eta)\, \mathbb{1}_{2\times 2} - \sinh(\eta)\, \ell$$

This we recognize as a boost in the x-direction!

$$\boxed{\Rightarrow \Lambda = \begin{bmatrix} \cosh(\eta) & -\sinh(\eta) & & \\ -\sinh(\eta) & \cosh(\eta) & & \\ & & 1 & \\ & & & 1 \end{bmatrix}}$$

The parameter η is nothing but the rapidity!

By similar calculations it is easy to show that J_1, J_2, J_3 indeed generate rotations.

For example, a rotation in the xy-plane using the parameter ω gives

$$L = +\omega \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \Rightarrow \quad \Lambda = \begin{bmatrix} 1 & & & \\ & \cos\omega & \sin\omega & \\ & -\sin\omega & \cos\omega & \\ & & & 1 \end{bmatrix}$$

sign in HUB p.153 is not consistent with def in eq. II.8.29, p.151

Comparing the above expressions for Λ we see that rapidities play the same role for boosts as angles do for rotations. They mix a spatial direction with time instead of mixing 2 spatial directions.

Ex: Tensor transformation for E- and B-fields.

We have seen that the electric and magnetic fields are part of an antisymmetric rank $\binom{2}{0}$- tensor:

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -cB_z & cB_y \\ E_y & cB_z & 0 & -cB_x \\ E_z & -cB_y & cB_x & 0 \end{bmatrix}$$

Warning: different books define $F^{\mu\nu}$ differently, for example with a factor c less or with a -.

Consider constant electric and magnetic fields and calculate how they are transformed in another system.

We may consider a boost in x-direction

$$\Lambda = \begin{bmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \qquad \Lambda^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu}$$

$\Lambda^0{}_0 = \Lambda^1{}_1 = \gamma$, $\Lambda^0{}_1 = \Lambda^1{}_0 = -\beta\gamma$, $\Lambda^2{}_2 = \Lambda^3{}_3 = 1$, all others are 0

A rank $\binom{2}{0}$ -tensor transforms as

$$F'^{\alpha\beta} = \frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x'^\beta}{\partial x^\nu} F^{\mu\nu} = \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu F^{\mu\nu}$$

<u>$E_\parallel = E_x$</u>

$$E_x' = F'^{10} = \underbrace{\Lambda^1{}_\mu}_{\Lambda^1{}_0,\ \Lambda^1{}_1} \underbrace{\Lambda^0{}_\nu}_{\Lambda^0{}_0,\ \Lambda^0{}_1} F^{\mu\nu}$$

$$= \Lambda^1{}_0 \Lambda^0{}_0 \underbrace{F^{00}}_{0} + \Lambda^1{}_0 \Lambda^0{}_1 F^{01} + \Lambda^1{}_1 \Lambda^0{}_0 F^{10} + \Lambda^1{}_1 \Lambda^0{}_1 \underbrace{F^{11}}_{0}$$

$$= (-\beta\gamma)(-\beta\gamma)(-E_x) + \gamma^2 E_x$$

$$= E_x \underbrace{\gamma^2(-\beta^2+1)}_{1} = E_x$$

The electric field <u>parallel</u> to the direction of the boost doesn't change!

$\underline{B_{\parallel}, B_x}$

$$B_x = \frac{F}{c}^{32}$$

$$B_x' = \frac{F}{c}'^{32} = \frac{1}{c} \underbrace{\Lambda^3{}_\mu}_{\Lambda^3{}_3 = 1} \underbrace{\Lambda^2{}_\nu}_{\Lambda^2{}_2 = 1} F^{\mu\nu} = \frac{1}{c} \Lambda^3{}_3 \Lambda^2{}_2 F^{32} = \frac{1}{c} F^{32}$$

$$\Rightarrow B_x' = B_x$$

B-field parallel to boost doesn't change!

$\underline{E_\perp, E_y}$

$$E_y = F^{20}$$

$$E_y' = F'^{20} = \underbrace{\Lambda^2{}_\mu}_{\Lambda^2{}_2} \underbrace{\Lambda^0{}_\nu}_{\Lambda^0{}_0, \Lambda^0{}_1} F^{\mu\nu}$$

$$= \Lambda^2{}_2 \Lambda^0{}_0 F^{20} + \Lambda^2{}_2 \Lambda^0{}_1 F^{21}$$

$$= 1 \cdot \gamma E_y + 1 (-\beta\gamma) c B_z$$

$$\Rightarrow E_y' = \gamma (E_y - \underbrace{v B_z}_{\text{contribution from magnetic field!}})$$

For the electric field <u>orthogonal</u> to the boost, the B-fields mix in!

$\underline{B_\perp, B_y}$

$$B_y = \frac{F}{c}^{13}$$

$$B_y' = \frac{F}{c}'^{13} = \underbrace{\Lambda^1{}_\mu}_{\Lambda^1{}_0, \Lambda^1{}_1} \underbrace{\Lambda^3{}_\nu}_{\Lambda^3{}_3} \frac{F}{c}^{\mu\nu}$$

$$= \Lambda^1{}_0 \Lambda^3{}_3 \frac{F}{c}^{03} + \Lambda^1{}_1 \Lambda^3{}_3 \frac{F}{c}^{13}$$

$$= (-\beta\gamma) \frac{F}{c}^{03} + \gamma \frac{F}{c}^{13}$$

$$= (-\beta\gamma)\left(-\frac{E_z}{c}\right) + \gamma B_y$$

$$= \gamma \left(B_y + \frac{v}{c^2} E_z\right)$$

For B-fields orthogonal to the direction of boost, the E-field mix in.

Warning: If E and B-fields depend on position we also have to transform the position to evaluate the fields in the right point, and if we need derivatives of the fields we need to take space contraction and time dilation into account.