Orthogonal multiplet bases in color space

In collaboration with
Stefan Keppeler (Tübingen),
arXiv:1207.0609, accepted by JHEP

- Motivation
- The standard method (trace basis)
- Orthogonal multiplet bases for SU(N_c)
- Conclusion and outlook

Lund
September 6, 2012
Malin Sjödahl

Motivation

- With the start of the LHC follows an increased demand of accurately calculated processes in QCD
- This is applicable to NLO calculations and resummation
- ...but my perspective is from a parton shower point of view
- First SU(3) parton shower in collaboration with Simon Plätzer JHEP 07(2012)042, color structure treated using my C++ ColorFull code

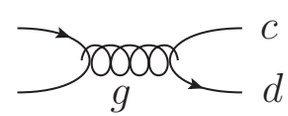
The color space

- We never observe individual colors $\rightarrow$ we are only interested in color summed quantities
- For given external partons, the color space is a finite dimensional vector space equipped with a scalar product

$$< A, B >= \sum_{a,b,c,\ldots} A_{a,b,c,\ldots}(B_{a,b,c,\ldots})^*$$

Example: If

$$A = \sum_g (t^g)^a{}_b (t^g)^c{}_d = \sum_g$$



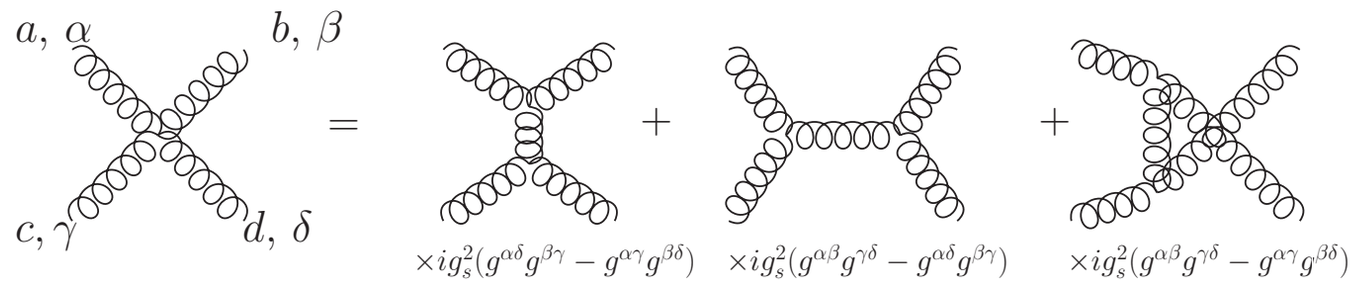
,

then $< A|A >= \sum_{a,b,c,d,g,h} (t^g)^a{}_b (t^g)^c{}_d (t^h)^b{}_a (t^h)^d{}_c$

- We may use any basis (spanning set)

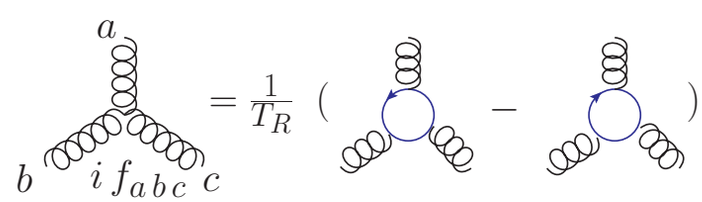
The standard treatment

- Every 4g vertex can be replaced by 3g vertices:

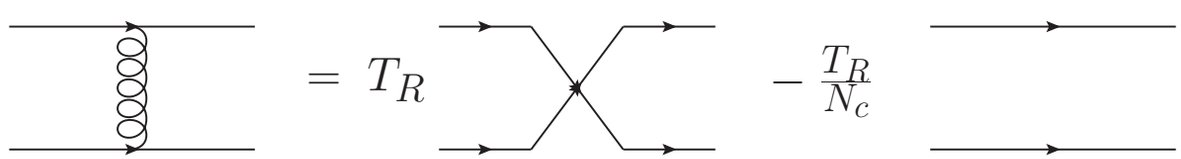


(read counter clockwise)

- Every 3g vertex can be replaced using:

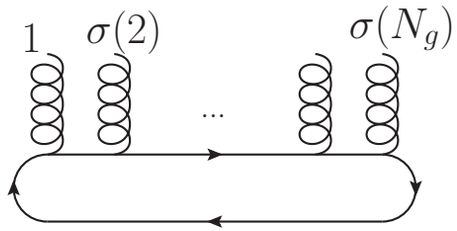


- After this every internal gluon can be removed using:



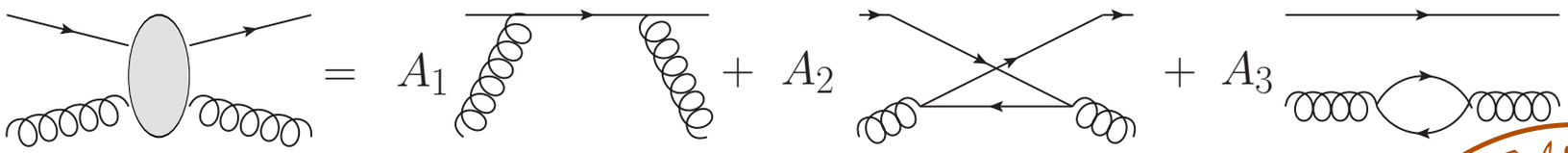
- This can be applied to any QCD amplitude, tree level or beyond
- For gluons at tree level, the result is a sum over closed quark-lines

$$A = \sum_{\sigma \in S_{N_g-1}} A_\sigma$$



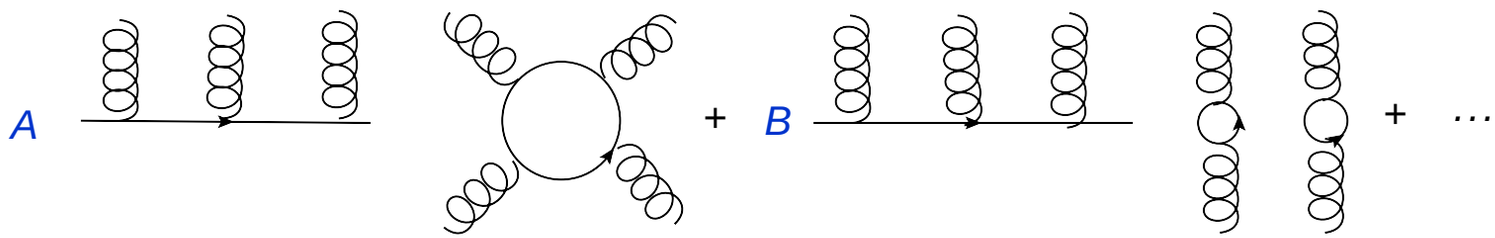
$$= \sum_{\sigma \in S_{N_g-1}} A_\sigma \mathrm{Tr}[t^1 t^{\sigma(2)} ... t^{\sigma(N_g)}]$$

- At one loop we may have a product of up to two traces, and for arbitrary order up to $N_g/2$ traces
- For processes with quarks there are open quark lines as well:
  For example for 2 (incoming + outgoing) gluons and one $q\bar{q}$ pair



(an incoming quark is the same as an outgoing anti-quark)

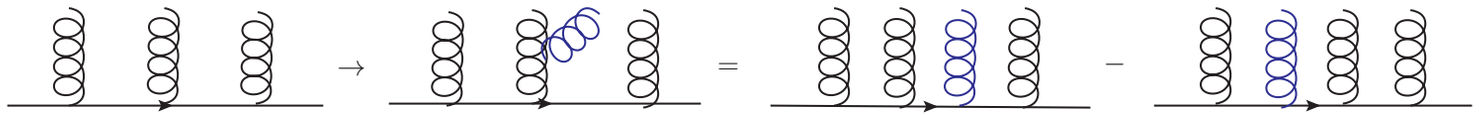
- In general an amplitude can be written as linear combination of different color structures, like



- This is the kind of "trace bases" used in ColorFull for the SU(3) parton shower, and in most NLO calculations

It has some nice properties

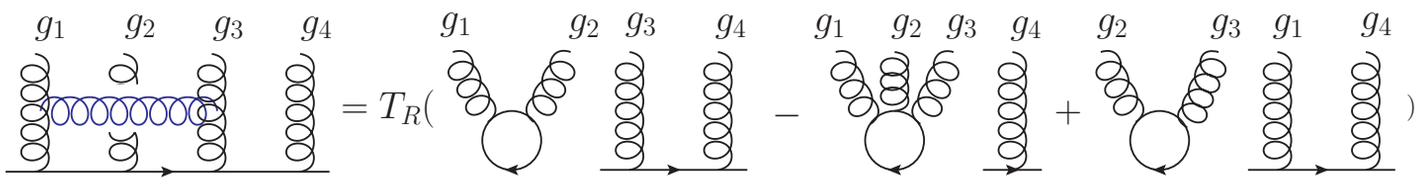
- The effect of gluon emission is easily described:



Convention: + when inserting after, minus when inserting before.

(Z. Nagy & D. Soper, JHEP 0807 (2008) 025)

- So is the effect of gluon exchange:



Convention: + when inserting after, - when inserting before

(M. Sjödahl, JHEP 0909 (2009) 087 JHEP)

ColorFull

For the purpose of treating a general QCD color structure I have written a C++ color algebra code, ColorFull, which:

- Is used in the color shower with Simon Plätzer
- automatically creates a "trace basis" for any number and kind of partons, and to any order in α_s
- describes the effect of gluon emission
- ... and gluon exchange
- squares color amplitudes
- can be used with boost for optimized calculations
- is planned to be published separately

However...

- This type of "basis" is non-orthogonal and overcomplete (for more than N_c gluons plus $q\bar{q}$-pairs)
- ... and the number of basis vectors grows as a factorial in $N_g + N_{q\bar{q}}$
  $\rightarrow$ when squaring amplitudes we run into a factorial square scaling
- Hard to go beyond $\sim$ 8 gluons plus $q\bar{q}$-pairs

However...

- This type of "basis" is non-orthogonal and overcomplete (for more than N_c gluons plus $q\bar{q}$-pairs)
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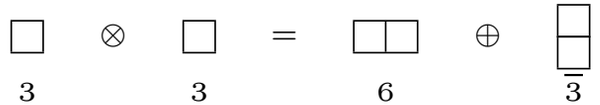
  $\rightarrow$ when squaring amplitudes we run into a factorial square scaling
- Hard to go beyond $\sim$ 8 gluons plus $q\bar{q}$-pairs
- Would be nice with minimal orthogonal basis

Orthogonal multiplet bases

In collaboration with Stefan Keppeler

- The color space may be decomposed into irreducible representations, enumerated using Young tableaux multiplication
- For example for $qq \to qq$ we have



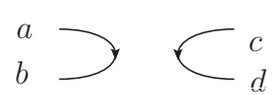
and the corresponding basis vectors

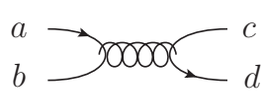
Here Cvitanović's birdtrack notation is used. These color tensors are orthogonal both when seen as qq-projectors, and when seen as basis vectors on the 4-parton space

- For quarks we can construct orthogonal projectors and basis vectors using Young tableaux ...at least from the Hermitian quark projectors
- In fact the $qq \to qq$ color space is the same as for $q\overline{q} \to q\overline{q}$,

$$\square \otimes \overline{\square} = \bullet \oplus \begin{smallmatrix}\square\square\\ \square\end{smallmatrix}$$

and we could as well have used the basis:

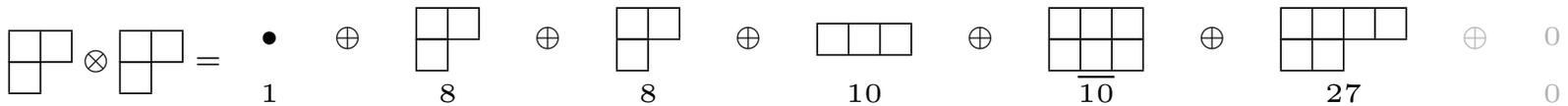
$\mathbf{V}^1 = \delta^a{}_b \delta^c{}_d =$ 

, $\mathbf{V}^8 = (t^g)^a{}_b (t^g)^c{}_d =$ 

- In general we may "comb" the involved particles as incoming and outgoing as we wish
- In QCD we have quarks, anti-quarks and gluons
  $\to$ No obvious way to construct projectors

The simplest gluon example, $gg \to gg$

- Basis vectors can be enumerated using Young tableaux multiplication



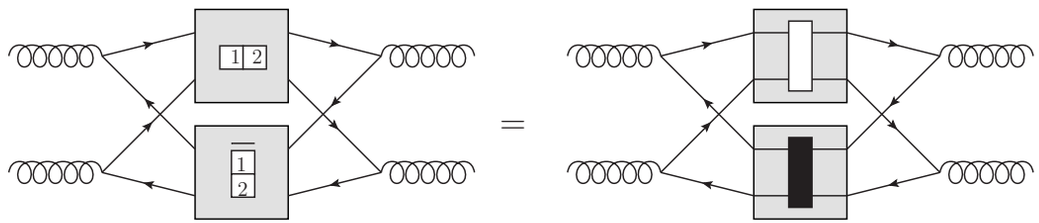
- As color is conserved an incoming multiplet of a certain kind can only go to an outgoing multiplet of the same kind, $1 \to 1, 8 \to 8$...

  Charge conjugation implies that some vectors only occur together...

The problem is the construction of the corresponding projection operators; the Young tableaux operate with "quark-units"

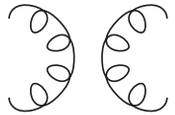
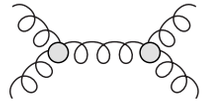
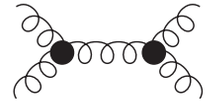
- Problem first solved for two gluons by MacFarlane, Sudbery, and Weisz 1968, however only for $N_c = 3$
- General N_c solution for two gluons by Cvitanović (in group theory books, 1984 and 2008), using polynomial equations
- General N_c solution for two gluons by Dokshitzer and Marchesini (2006), using symmetries and intelligent guesswork

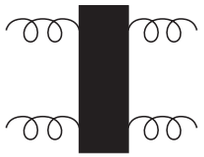
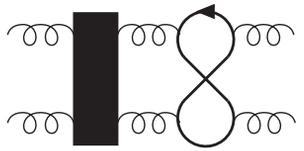
- For two gluons, there are two octet projectors, one singlet projector, and 4 new projectors, 10, $\overline{10}, 27$, and for general N_c, "0"
- It turns out that the new projectors can be seen as corresponding to different symmetries w.r.t. quark and anti-quark units, for example the decuplet can be seen as corresponding to

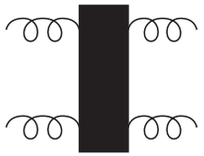
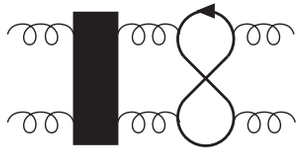


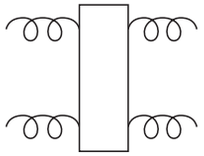
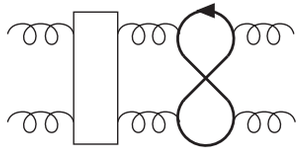
– (singlet) and octet(s) – (singlet) and octet(s)

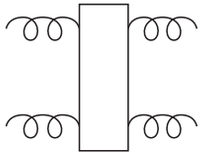
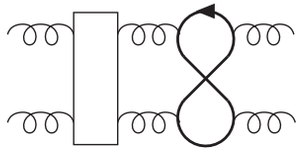
Similarly the anti-decuplet corresponds to $\begin{smallmatrix}1\\2\end{smallmatrix} \otimes \overline{\begin{smallmatrix}1&2\end{smallmatrix}}$, the 27-plet corresponds to $\begin{smallmatrix}1&2\end{smallmatrix} \otimes \overline{\begin{smallmatrix}1&2\end{smallmatrix}}$ and the 0-plet to $\begin{smallmatrix}1\\2\end{smallmatrix} \otimes \overline{\begin{smallmatrix}1\\2\end{smallmatrix}}$

$$\mathbf{P}^{1} = \frac{1}{N_c^2-1}$$  $$\mathbf{P}^{8s} = \frac{N_c}{2T_R(N_c^2-4)}$$  $$\mathbf{P}^{8a} = \frac{1}{2N_cT_R}$$ 

$$\mathbf{P}^{10} = \frac{1}{2}$$  $$+\frac{1}{2T_R^2}$$  $$-\frac{1}{2}\mathbf{P}^{8a}$$

$$\mathbf{P}^{\overline{10}} = \frac{1}{2}$$  $$-\frac{1}{2T_R^2}$$  $$-\frac{1}{2}\mathbf{P}^{8a}$$

$$\mathbf{P}^{27} = \frac{1}{2}$$  $$+\frac{1}{2T_R^2}$$  $$-\frac{N_c-2}{2N_c}\mathbf{P}^{8s}-\frac{N_c-1}{2N_c}\mathbf{P}^{1}$$

$$\mathbf{P}^{0} = \frac{1}{2}$$  $$-\frac{1}{2T_R^2}$$  $$-\frac{N_c+2}{2N_c}\mathbf{P}^{8s}-\frac{N_c+1}{2N_c}\mathbf{P}^{1}$$

New idea: Could this work in general?

On the one hand side

$$g_1 \otimes g_2 \otimes .... \otimes g_n \subseteq (q_1 \otimes \bar{q}_1) \otimes (q_2 \otimes \bar{q}_2) \otimes ... \otimes (q_n \otimes \bar{q}_n)$$

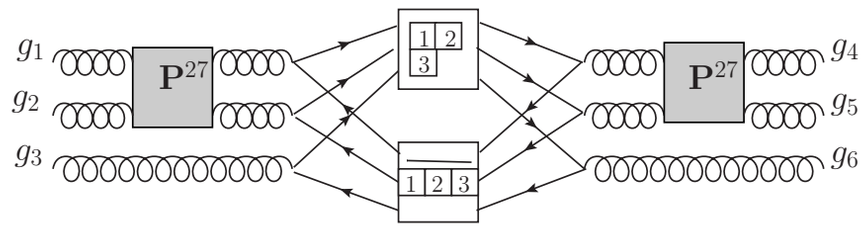
so there is hope...

On the other hand...

- Why should it?
- In general there are many instances of a multiplet, how do we know we construct all?
- Even if such a decomposition would give the new multiplets (which could not be present for fewer gluons) in a unique way, we would have to project out all instances of all "old" multiplets. How do we get them?

Key observation:

- Starting in a given multiplet, corresponding to some $q\overline{q}$ symmetries, such as 27, from $\boxed{1}\boxed{2} \otimes \overline{\boxed{1}\boxed{2}}$, it turns out that for each way of attaching a quark box to $\boxed{1}\boxed{2}$ and an anti-quark box to $\overline{\boxed{1}\boxed{2}}$, to there is at most one new multiplet! For example, the projector $\mathbf{P}^{27,\overline{35}}$ can be seen as coming from

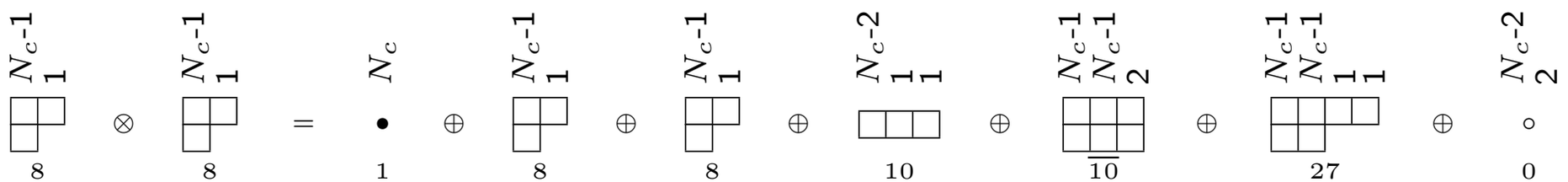


after having projected out "old" multiplets

- In fact, for large enough N_c, there is precisely one new multiplet for each set of $q\overline{q}$ symmetries

It turns out that the proof of this is really interesting:

- We find that the irreducible representations in $g^{\otimes n_g}$ for varying N_c stand in a one to one, or one to zero correspondence to each other! (For each SU(3) multiplet there is an SU(5) version, but not vice versa.)
- Every multiplet in $g^{\otimes n_g}$ can be labeled in an N_c-independent way using the lengths of the *columns*. For example



I have not seen this column notation elsewhere... have you?

Projecting out "old" multiplets

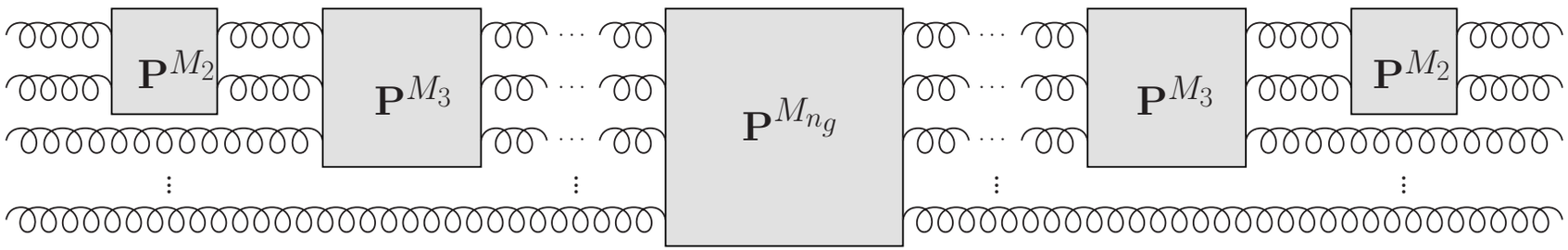
This would give us a way of constructing all projectors corresponding to "new" multiplets, if we knew how to project out all old multiplets.

In $g_1 \otimes g_2 \otimes g_3$, there are many 27-plets. How do we separate the various instance of the same multiplet?

Projecting out "old" multiplets

This would give us a way of constructing all projectors corresponding to "new" multiplets, if we knew how to project out all old multiplets. In $g_1 \otimes g_2 \otimes g_3$, there are many 27-plets. How do we separate the various instance of the same multiplet?

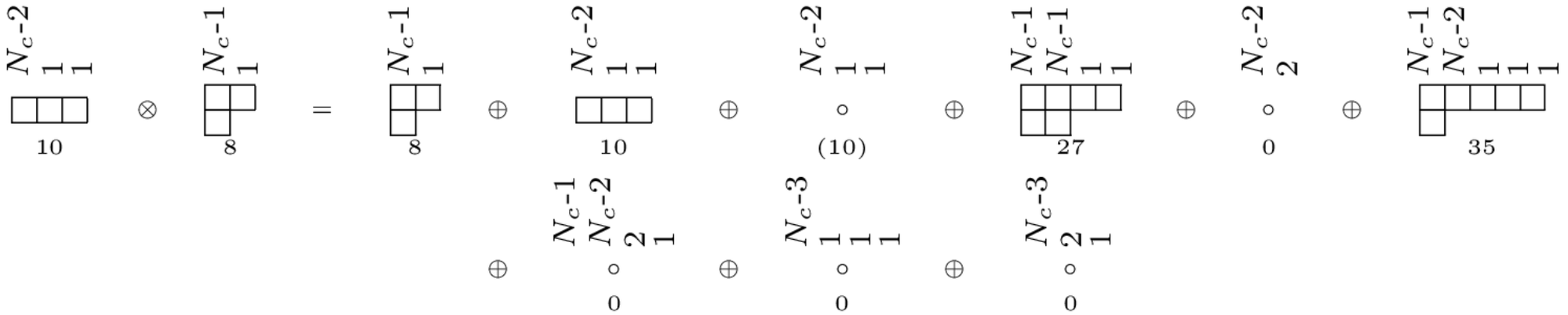
- By the construction history!

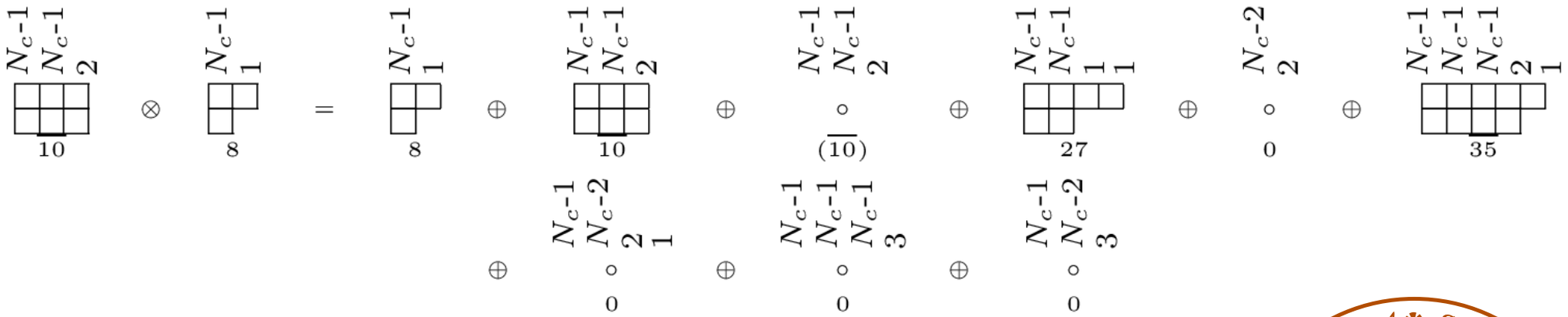


We make sure that the $n_g - \nu$ first gluons are in a given multiplet! Then the various instances are orthogonal as, at some point in the construction history, there was a different projector!

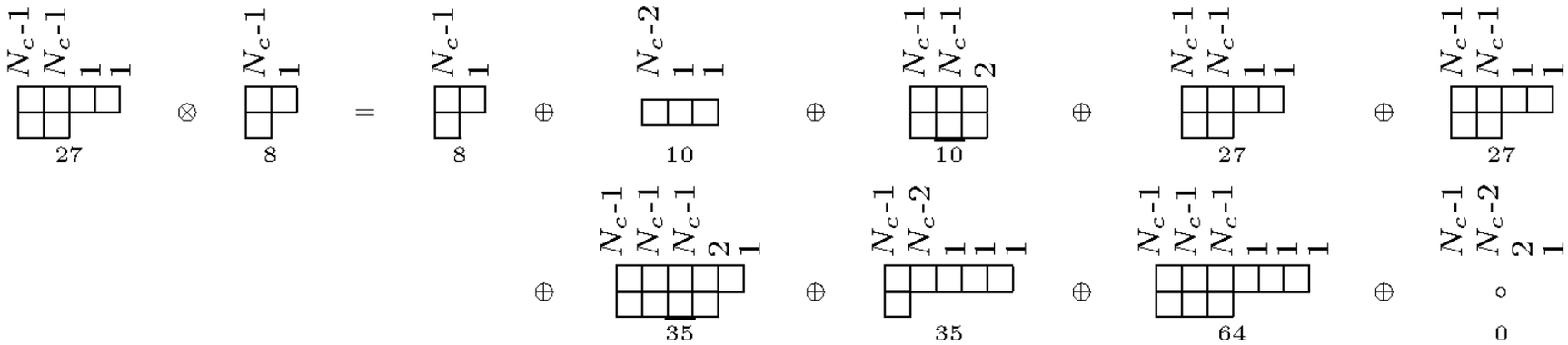
(More complicated for multiple occurrences...)

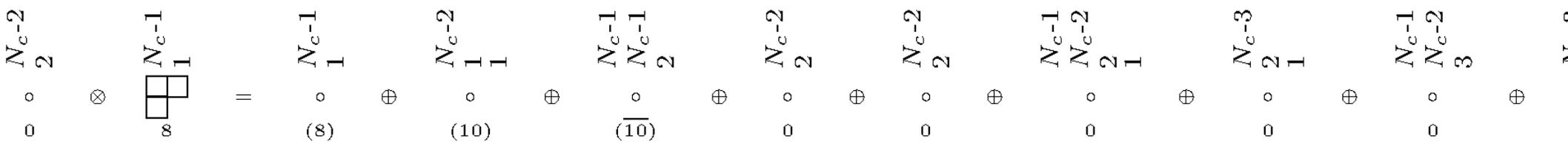
The 3g multiplets from (anti-) decuplets



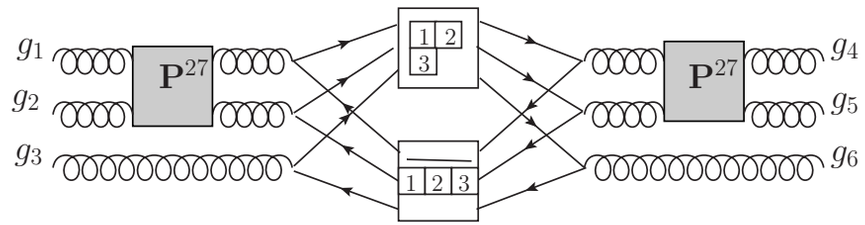


The 3g multiplets from 27- and 0-plets





- Construct projectors corresponding to "old" multiplets
- Construct the tensors which will give rise to "new" projectors



- From these, project out "old" multiplets

- Calculations are done using a mathematica package, ColorMath
- Separate publication planned this autumn
- Intended to be an easy to use mathematica package for color summed calculations in QCD, $SU(N_c)$

```
In[1]:= Get["/data/Documents/Annatjobb/Color/Mathematica/ColorMathv5.m"]

In[2]:= Amplitude = T t{g}q1q3 t{g}q4q2 + S t{g}q1q2 t{g}q4q3;

In[3]:= Amplitude Conjugate[Amplitude /. g → g2] // CSimplify // Simplify
```

Out[3]= $$\frac{\left(-1+N_c^2\right)\left(\text{Conjugate}[S]\left(-T+S\,N_c\right)+\text{Conjugate}[T]\left(-S+T\,N_c\right)\right)T_F^2}{N_c}$$

- In this way we have constructed the projection operators onto irreducible subspaces for $3g \to 3g$
- There are 51 projectors, reducing to 29 for SU(3)
- From these we have constructed an orthogonal (normalized) basis for the $6g$ space, by letting any instance of a given multiplet go to any other instance of the same multiplet. For general N_c there are 265 basis vectors. Crossing out tensors that do not appear for $N_c = 3$, we get a minimal basis with 145 basis vectors.

There's also a reduction from charge conjugation

Number of projection operators and basis vectors

In general, for many partons the size of the vector space is much smaller for $N_c = 3$, compared to for $N_c \to \infty$

Case	Projectors $N_c = 3$	Projectors $N_c = \infty$	Vectors $N_c = 3$	Vectors $N_c = \infty$
2g $\to$ 2g	6	7	8	9
3g $\to$ 3g	29	51	145	265
4g $\to$ 4g	166	513	3 598	14 833
5g $\to$ 5g	1 002	6 345	107 160	1 334 961

Number of projection operators and basis vectors for $N_g \to N_g$ gluons *without* imposing projection operators and vectors to appear in charge conjugation invariant combinations

- The size of the vector spaces asymptotically grows as an exponential in the number of gluons/$q\bar{q}$-pairs for finite N_c
- For general N_c the basis size grows as a factorial

$$N_{\text{vec}}[n_q, N_g] = N_{\text{vec}}[n_q, N_g - 1](N_g - 1 + n_q) + N_{\text{vec}}[n_q, N_g - 2](N_g - 1)$$

where

$$\begin{aligned} N_{\text{vec}}[n_q, 0] &= n_q! \\ N_{\text{vec}}[n_q, 1] &= n_q n_q! \end{aligned}$$

As the multiplet basis also is orthogonal it has the potential to very significantly speed up exact calculations in QCD!

Processes with quarks

- We can also construct bases for processes with quarks using the gluon projection operators. To see this we note that a $q\overline{q}$-pair may either be in an octet – in which case we may replace it with a gluon, or in a singlet – in which case we enforce this and use the gluon basis for one less gluon
- In general, having the $n_g \to n_g$ projectors we can easily get the bases for up to $2n_g + 1$ gluons plus $q\overline{q}$ pairs
- Knowing how to construct the gluon projection operators in general, we thus know how to construct the basis vectors for any number and kind of partons and any order in perturbation theory!

Conclusions

- We have outlined a general recipe for construction of minimal orthogonal multiplet based bases for any QCD process, arXiv:1207.0609
- On the way we found an N_c-independent labeling of the multiplets in $g^{\otimes n_g}$, and a one to one, or one to zero, correspondence between these for various N_c
- Number of basis vectors grows only exponentially for $N_c = 3$
- This has the potential to very significantly speed up exact calculations in the color space of $SU(N_c)$

...and outlook

- However, in order to use this in an optimized way, we need to understand how to sort QCD amplitudes in this basis in an efficient way
- ...also, a lot of implementational work remains

Backup: Some example projectors

$$\mathbf{P}^{8a,8a}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} = \frac{1}{T_R^2}\frac{1}{4N_c^2} i f_{g_1\,g_2\,i_1} i f_{i_1\,g_3\,i_2} i f_{g_4\,g_5\,i_3} i f_{i_3\,g_6\,i_2}$$

$$\mathbf{P}^{8s,27}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} = \frac{1}{T_R}\frac{N_c}{2(N_c^2-4)} d_{g_1\,g_2\,i_1} \mathbf{P}^{27}_{i_1\,g_3\,i_2\,g_6} d_{i_2\,g_4\,g_5}$$

$$\mathbf{P}^{27,8}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} = \frac{4(N_c+1)}{N_c^2(N_c+3)} \mathbf{P}^{27}_{g_1\,g_2\,i_1\,g_3} \mathbf{P}^{27}_{i_1\,g_6\,g_4\,g_5}$$

$$\mathbf{P}^{27,64=c111c111}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} = \frac{1}{T_R^3}\mathbf{T}^{27,64}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} - \frac{N_c^2}{162(N_c+1)(N_c+2)}\mathbf{P}^{27,8}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6}$$

$$- \frac{N_c^2-N_c-2}{81N_c\,(N_c+2)}\mathbf{P}^{27,27s}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6}$$

Backup: Three gluon multiplets

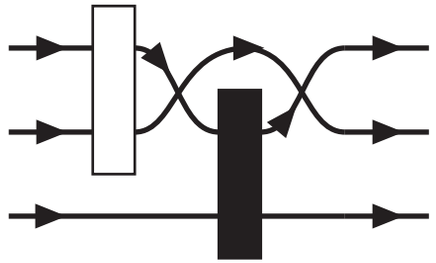
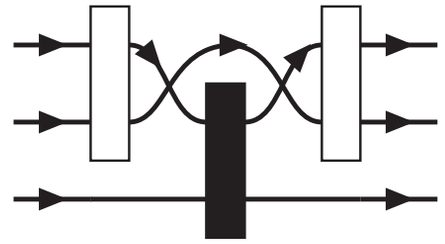
$SU(3)$ dim	1	8	10	$\overline{10}$	27	0
Multiplet	c0c0	c1c1	c11c2	c2c11	c11c11	c2c2
	$((45)^{8s}6)^{1}$	$2 \times ((45)^{8s}6)^{8s \text{ or } a}$	$((45)^{8s}6)^{10}$	$((45)^{8s}6)^{\overline{10}}$	$((45)^{8s}6)^{27}$	$((45)^{8s}6)^{0}$
	$((45)^{8a}6)^{1}$	$2 \times ((45)^{8a}6)^{8s \text{ or } a}$	$((45)^{8a}6)^{10}$	$((45)^{8a}6)^{\overline{10}}$	$((45)^{8a}6)^{27}$	$((45)^{8a}6)^{0}$
		$((45)^{10}6)^{8}$	$((45)^{10}6)^{10}$	$((45)^{\overline{10}}6)^{\overline{10}}$	$((45)^{10}6)^{27}$	$((45)^{10}6)^{0}$
		$((45)^{\overline{10}}6)^{8}$	$((45)^{10}6)^{10}$	$((45)^{\overline{10}}6)^{\overline{10}}$	$((45)^{\overline{10}}6)^{27}$	$((45)^{\overline{10}}6)^{0}$
		$((45)^{27}6)^{8}$	$((45)^{27}6)^{10}$	$((45)^{27}6)^{\overline{10}}$	$((45)^{27}6)^{27}$	$((45)^{0}6)^{0}$
		$((45)^{0}6)^{8}$	$((45)^{0}6)^{10}$	$((45)^{0}6)^{\overline{10}}$	$((45)^{27}6)^{27}$	$((45)^{0}6)^{0}$

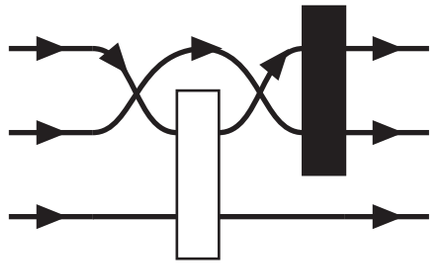
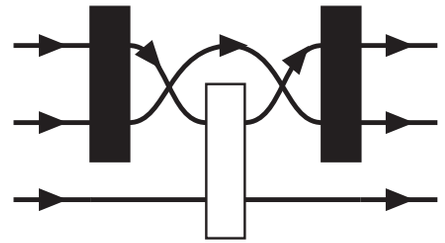
$SU(3)$ dim	64	35	$\overline{35}$	0
Multiplet	c111c111	c111c21	c21c111	c21c21
	$((45)^{27}6)^{64}$	$((45)^{10}6)^{35}$	$((45)^{\overline{10}}6)^{\overline{35}}$	$((45)^{10}6)^{c21c21}$
		$((45)^{27}6)^{35}$	$((45)^{27}6)^{\overline{35}}$	$((45)^{\overline{10}}6)^{c21c21}$
				$((45)^{27}6)^{c21c21}$
				$((45)^{0}6)^{c21c21}$

$SU(3)$ dim	0	0	0	0	0
Multiplet	c111c3	c3c111	c21c3	c3c21	c3c3
	$((45)^{10}6)^{c111c3}$	$((45)^{\overline{10}}6)^{c3c111}$	$((45)^{10}6)^{c21c3}$	$((45)^{\overline{10}}6)^{c3c21}$	$((45)^{0}6)^{c3c3}$
			$((45)^{0}6)^{c21c3}$	$((45)^{0}6)^{c3c21}$	

Multiplets for $g_4 \otimes g_5 \otimes g_6$

The importance of Hermitian projectors

$\mathbf{P}_Y^{6,8} = \frac{4}{3}$  , $\mathbf{P}_Y^{6,8} = \frac{4}{3}$ 

$\mathbf{P}_Y^{\bar{3},8} = \frac{4}{3}$  , $\mathbf{P}^{\bar{3},8} = \frac{4}{3}$ 

The standard Young projection operators $\mathbf{P}_Y^{6,8}$ and $\mathbf{P}_Y^{\bar{3},8}$ compared to their hermitian versions $\mathbf{P}^{6,8}$ and $\mathbf{P}^{\bar{3},8}$.
Clearly $\mathbf{P}^{6,8\dagger}\mathbf{P}^{\bar{3},8} = \mathbf{P}^{6,8}\mathbf{P}^{\bar{3},8} = 0$. However, as can be seen from the symmetries, $\mathbf{P}_Y^{6,8\dagger}\mathbf{P}_Y^{\bar{3},8} \neq 0$.

Backup: First occurrence

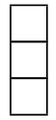
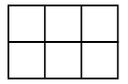
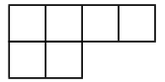
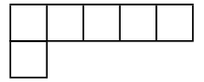
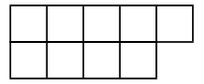
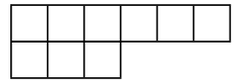
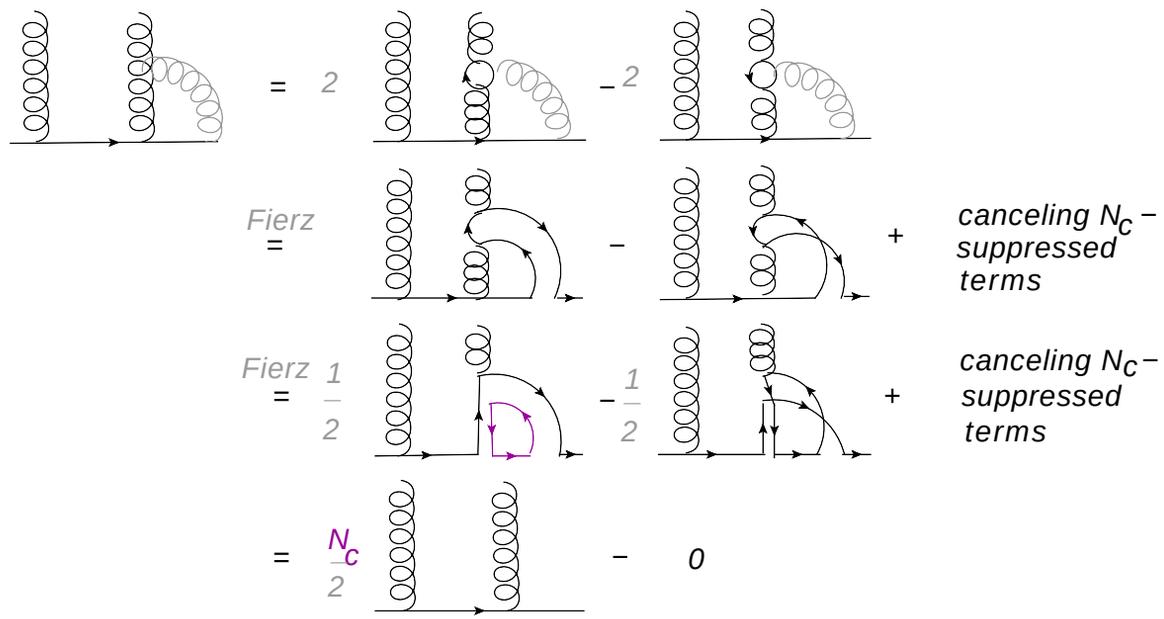
n_f	0	1	2	3
$SU(3)$ Young diagrams	$\bullet =$ 		  	  

Table 1: Examples of $SU(3)$ Young diagrams sorted according to their first occurrence n_f.

Backup: Gluon exchange

A gluon exchange in this basis "directly" i.e. without using scalar products gives back a linear combination of (at most 4) basis tensors



- N_c-enhancement possible only for near by partons
  $\rightarrow$ only "color neighbors" radiate in the $N_c \rightarrow \infty$ limit