Tools for calculations in color space

- Dealing with exact color summed calculations
- Theoretical tools:
  - "Trace bases"
  - Orthogonal multiplet bases
- Computational tools:
  - ColorFull C++
  - ColorMath Mathematica
- Conclusions

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Motivation

- With the LHC follows an increased demand of accurately calculated processes in QCD
- This is applicable to NLO calculations and resummation
- ...but my perspective is from a parton shower point of view
- First SU(3) parton shower in collaboration with Simon Plätzer JHEP 07(2012)042, arXiv:1201.0260 color structure treated using my C++ ColorFull code

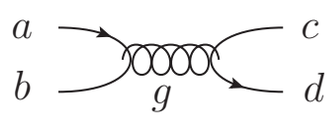
Dealing with color space

- We never observe individual colors
  $\rightarrow$ we are only interested in color summed/averaged quantities
- For given external partons, the color space is a finite dimensional vector space equipped with a scalar product

$$\langle A, B\rangle = \sum_{a,b,c,\ldots} (A_{a,b,c,\ldots})^* B_{a,b,c,\ldots}$$

Example: If

$$A = \sum_g (t^g)^a{}_b (t^g)^c{}_d =$$



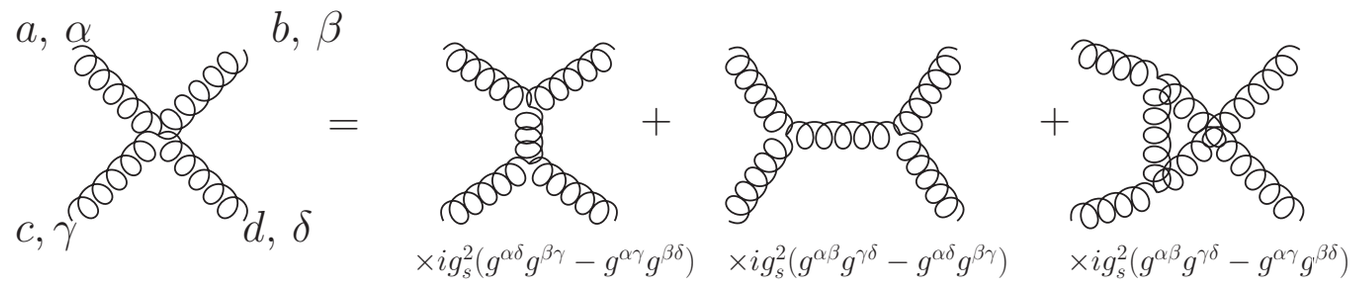
,

then $\langle A|A\rangle = \sum_{a,b,c,d,g,h} (t^h)^b{}_a (t^h)^d{}_c (t^g)^a{}_b (t^g)^c{}_d$

- One way of dealing with color space is to just square the amplitudes as one encounters them
- Alternatively, we may use any basis (spanning set)

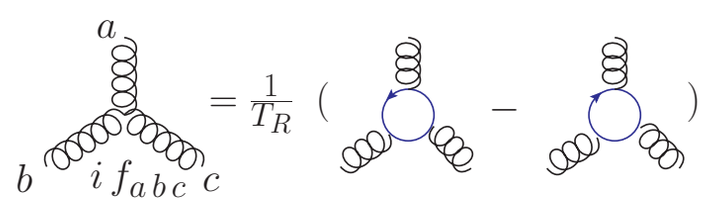
The standard treatment: Trace bases

- Every 4g vertex can be replaced by 3g vertices:

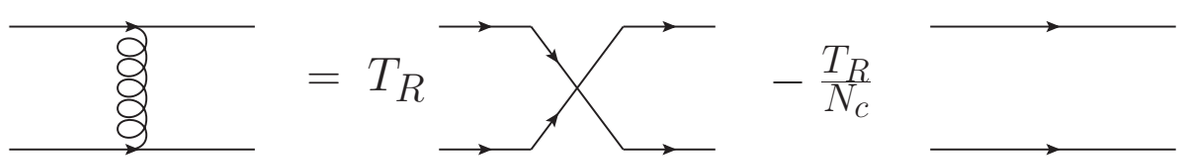


(read counter clockwise)

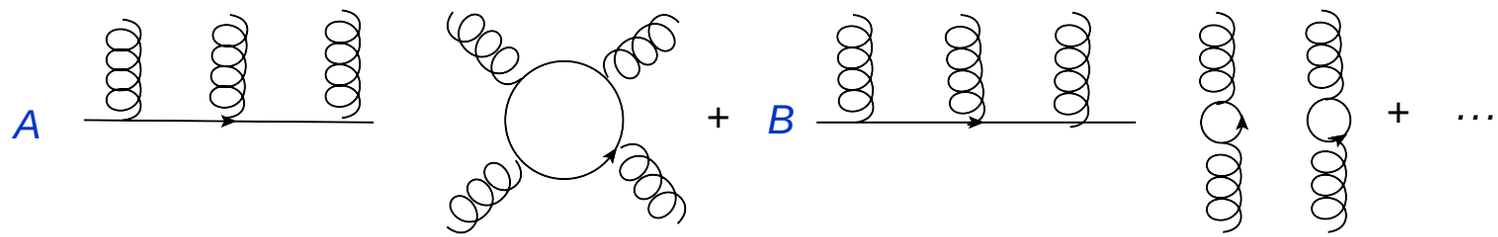
- Every 3g vertex can be replaced using:



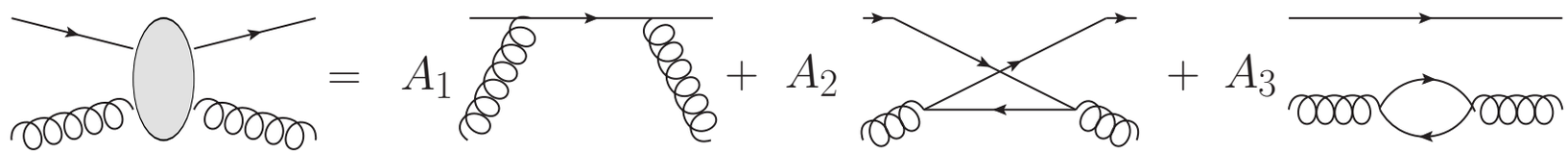
- After this every internal gluon can be removed using:



- This can be applied to any QCD amplitude, tree level or beyond
- In general an amplitude can be written as linear combination of different color structures, like



- For example for 2 (incoming + outgoing) gluons and one $q\bar{q}$ pair

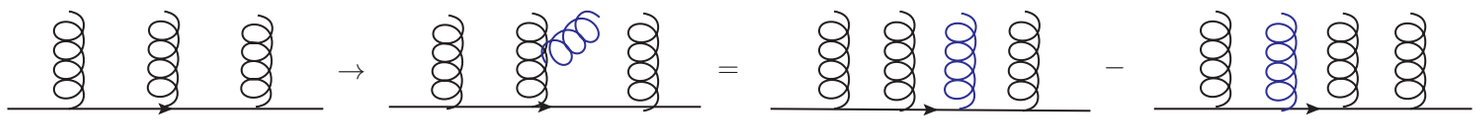


(an incoming quark is the same as an outgoing anti-quark)

The above type of color structures can be used as a spanning set, a trace basis. (Technically it's in general overcomplete, so it is rather a spanning set.)

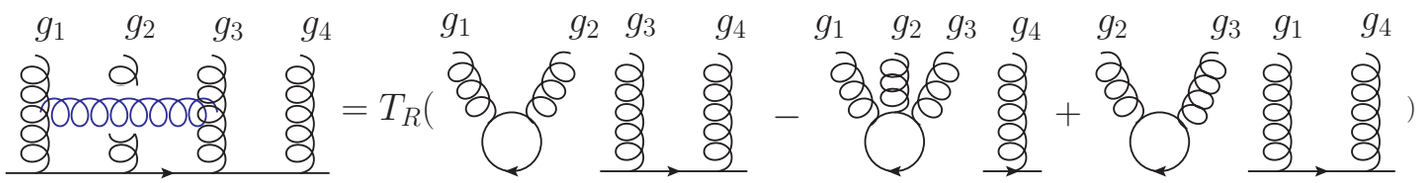
These bases have some nice properties

- The effect of gluon emission is easily described:



Convention: + when inserting after, minus when inserting before.

- So is the effect of gluon exchange:



Convention: + when inserting after, - when inserting before

ColorFull

For the purpose of treating a general QCD color structure I have written a C++ color algebra code, ColorFull, which:

- Is used in the color shower with Simon Plätzer
- Collaborates with Simon's Matchbox code
- Automatically creates a "trace basis" for any number and kind of partons, and to any order in α_s
- Describes the effect of gluon emission
- ... and gluon exchange
- Squares color amplitudes
- Is planned to be published separately later this year

However...

- This type of "basis" is non-orthogonal and overcomplete (for more than N_c gluons plus $q\bar{q}$-pairs)
- ... and the number of spanning vectors grows as a factorial in $N_g + N_{q\bar{q}}$
  $\rightarrow$ when squaring amplitudes we run into a factorial square scaling
- Hard to go beyond $\sim$ 8 gluons plus $q\bar{q}$-pairs

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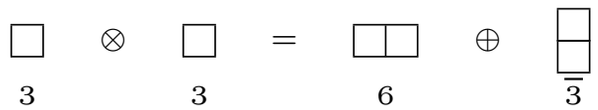
- Would be nice with minimal orthogonal basis

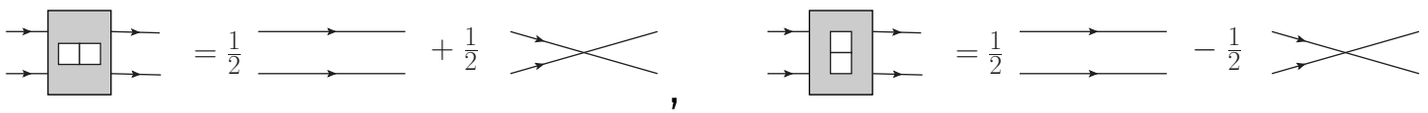
Orthogonal multiplet bases

In collaboration with Stefan Keppeler (Tübingen)

- QCD is based on SU(3) $\rightarrow$ the color space may be decomposed into irreducible representations, enumerated using Young tableau multiplication
- For example for $qq \rightarrow qq$ we have



and the corresponding basis vectors

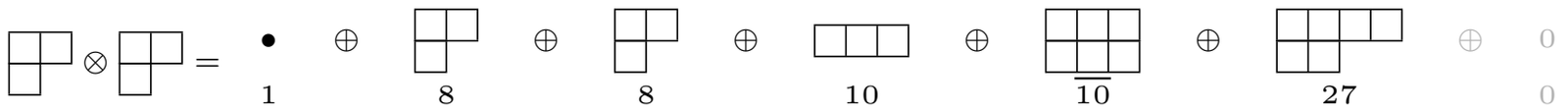


These color tensors are orthogonal both when seen as qq-projectors, and when seen as basis vectors on the 4-parton space

- For quarks we can construct orthogonal projectors and basis vectors using Young tableaux ...at least from the Hermitian quark projectors
- An incoming anti-quark may be treated as an outgoing quark
- In general we may "comb" the involved particles as incoming and outgoing as we wish
  $\rightarrow$ no problem to deal with any number of quarks and anti-quarks
- In QCD we have quarks, anti-quarks and gluons

Dealing with gluons

- Consider $gg \to gg$, the basis vectors can be enumerated using Young tableaux multiplication



- As color is conserved an incoming multiplet of a certain kind can only go to an outgoing multiplet of the same kind, $1 \to 1, 8 \to 8...$ $\to$ We know what to expect (Charge conjugation implies that some vectors only occur together)
- The problem is the construction of the corresponding projection operators; the Young tableaux operate with "quark-units" we need to deal also with gluons

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- However, until recently only a few cases had been dealt with, those for which (loosely speaking) nothing more complicated than two gluon projection operators is needed
- About one year ago me and Stefan Keppeler presented a general recipe for constructing gluon projection operators JHEP09(2012)124, arXiv:1207.0609

- Using these we can find orthogonal minimal multiplet bases for any number of gluons
- From these we can construct *orthogonal minimal* bases for any number of quarks and gluons and any N_c
- We have explicitly constructed orthogonal $3g \to 3g$ projectors and the corresponding six gluon orthogonal bases

JHEP09(2012)124, arXiv:1207.0609

- For many partons the size of the vector space is much smaller for $N_c = 3$ (exponential), compared to for $N_c \to \infty$ (factorial)

Case	Vectors $N_c = 3$	Vectors, general case
4 gluons	8	9
6 gluons	145	265
8 gluons	3 598	14 833
10 gluons	107 160	1 334 961

Number of basis vectors for $N_g \to N_g$ gluons *without* imposing vectors to appear in charge conjugation invariant combinations

- Multiplet bases have the potential to very significantly speed up exact calculations

ColorMath

- Calculations are done using my Mathematica package, ColorMath, Eur. Phys. J. C 73:2310 (2013), arXiv:1211.2099
- ColorMath is an easy to use Mathematica package for color summed calculations in QCD, SU(N_c)
- Repeated indices are implicitly summed

```
In[2]:= Amplitude = I f[g1, g2, g] t[{g}, q1, q2]
```

Out[2]= $i\, t^{\{g\}\,q1}{}_{q2}\, f^{\{g1,g2,g\}}$

```
In[3]:= CSimplify[Amplitude Conjugate[Amplitude /. g → h]]
```

Out[3]= $2\,Nc\,\left(-1+Nc^2\right)\,TR^2$

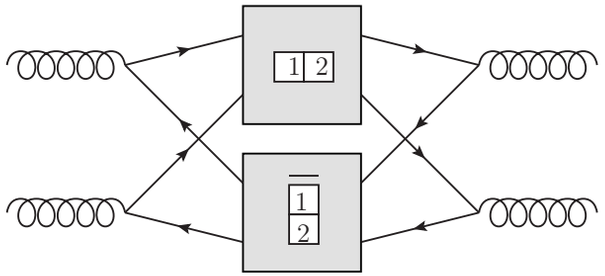
- The package and tutorial can be downloaded from `http://library.wolfram.com/infocenter/MathSource/8442/` or `www.thep.lu.se/~malin/ColorMath.html`

Conclusions

- One way of dealing with color space is to use "trace bases"
- This method is pursued in my and Simon Plätzer's $N_c = 3$ parton shower (JHEP 07(2012)042, arXiv:1201.0260) and in ColorFull
- This type of basis is not orthogonal and not minimal
- With Stefan Keppeler I have outlined a general recipe for construction of minimal orthogonal multiplet based bases for any QCD process (JHEP09(2012)124, arXiv:1207.0609)
- This has the potential to very significantly speed up exact calculations in the color space of SU(N_c)
- I have also written a Mathematica package ColorMath for performing color summed calculations in SU(N_c) (Eur. Phys. J. C 73:2310 (2013), arXiv:1211.2099)

Backup: 2 gluon solutions

- For two gluons, there are two octet projectors, one singlet projector, and 4 "new" projectors, 10, $\overline{10}, 27$, and for general N_c, "0"
- It turns out that the new projectors can be seen as corresponding to different symmetries w.r.t. quark and anti-quark units, for example the decuplet can be seen as corresponding to

$\mathbf{P}^{10} \propto$  $-$ octet(s) $-$ (singlet)

Similarly the anti-decuplet corresponds to $\boxed{\begin{smallmatrix}1\\2\end{smallmatrix}} \otimes \overline{\boxed{1\,2}}$, the 27-plet corresponds to $\boxed{1\,2} \otimes \overline{\boxed{1\,2}}$ and the 0-plet to $\boxed{\begin{smallmatrix}1\\2\end{smallmatrix}} \otimes \overline{\boxed{\begin{smallmatrix}1\\2\end{smallmatrix}}}$

Backup: 2 gluon projectors

- Problem first solved for two gluons by MacFarlane, Sudbery, and Weisz 1968, however only for $N_c = 3$
- General N_c solution for two gluons by Butera, Cicuta and Enriotti 1979
- General N_c solution for two gluons by Cvitanović, in group theory books, 1984 and 2008, using polynomial equations
- General N_c solution for two gluons by Dokshitzer and Marchesini 2006, using symmetries and intelligent guesswork

Backup: Could this work in general?

On the one hand side

$$g_1 \otimes g_2 \otimes .... \otimes g_n \subseteq (q_1 \otimes \bar{q}_1) \otimes (q_2 \otimes \bar{q}_2) \otimes ... \otimes (q_n \otimes \bar{q}_n)$$

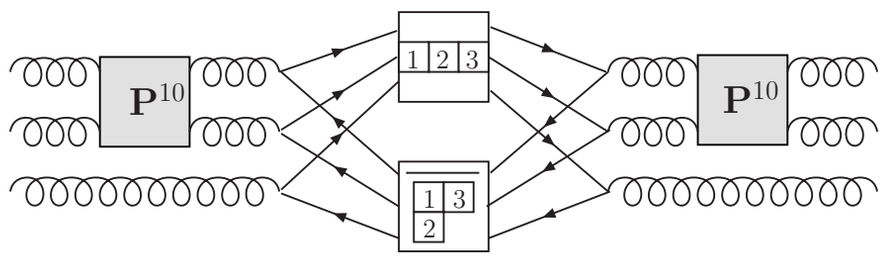
so there is hope...

On the other hand...

- Why should it?
- In general there are many instances of a multiplet, how do we know we construct all?

Backup: Key observation:

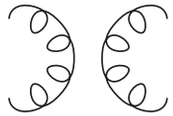
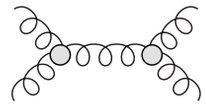
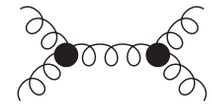
- Starting in a given multiplet, corresponding to some $q\overline{q}$ symmetries, such as 10, from $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \otimes \overline{\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}}$, it turns out that for each way of attaching a quark box to $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}$ and an anti-quark box to $\overline{\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}}$, to there is at most one new multiplet! For example, the projector $\mathbf{P}^{10,35}$ can be seen as coming from

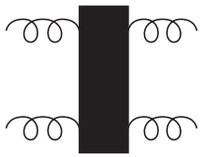
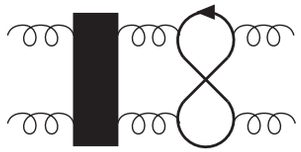


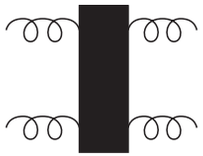
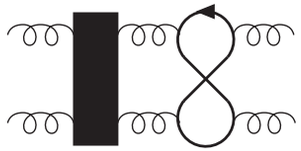
after having projected out "old" multiplets

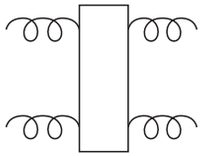
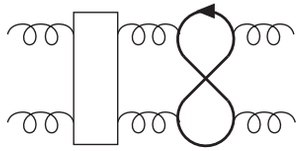
- In fact, for large enough N_c, there is precisely one new multiplet for each set of $q\overline{q}$ symmetries

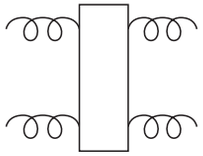
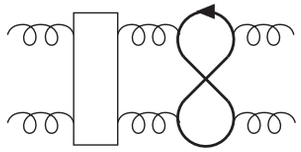
Backup: 2 gluon projectors

$\mathbf{P}^{1} = \frac{1}{N_c^2-1}$  , $\mathbf{P}^{8s} = \frac{N_c}{2T_R(N_c^2-4)}$  , $\mathbf{P}^{8a} = \frac{1}{2N_cT_R}$  ,

$\mathbf{P}^{10} = \frac{1}{2}$  $+ \frac{1}{2T_R^2}$  $- \frac{1}{2}\mathbf{P}^{8a}$

$\mathbf{P}^{\overline{10}} = \frac{1}{2}$  $- \frac{1}{2T_R^2}$  $- \frac{1}{2}\mathbf{P}^{8a}$

$\mathbf{P}^{27} = \frac{1}{2}$  $+ \frac{1}{2T_R^2}$  $- \frac{N_c-2}{2N_c}\mathbf{P}^{8s} - \frac{N_c-1}{2N_c}\mathbf{P}^{1}$

$\mathbf{P}^{0} = \frac{1}{2}$  $- \frac{1}{2T_R^2}$  $- \frac{N_c+2}{2N_c}\mathbf{P}^{8s} - \frac{N_c+1}{2N_c}\mathbf{P}^{1}$

Backup: Some 3g example projectors

$$\mathbf{P}^{8a,8a}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} = \frac{1}{T_R^2}\frac{1}{4N_c^2} if_{g_1\,g_2\,i_1} if_{i_1\,g_3\,i_2} if_{g_4\,g_5\,i_3} if_{i_3\,g_6\,i_2}$$

$$\mathbf{P}^{8s,27}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} = \frac{1}{T_R}\frac{N_c}{2(N_c^2-4)} d_{g_1\,g_2\,i_1}\mathbf{P}^{27}_{i_1\,g_3\,i_2\,g_6} d_{i_2\,g_4\,g_5}$$

$$\mathbf{P}^{27,8}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} = \frac{4(N_c+1)}{N_c^2(N_c+3)}\mathbf{P}^{27}_{g_1\,g_2\,i_1\,g_3}\mathbf{P}^{27}_{i_1\,g_6\,g_4\,g_5}$$

$$\mathbf{P}^{27,64=c111c111}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} = \frac{1}{T_R^3}\mathbf{T}^{27,64}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} - \frac{N_c^2}{162(N_c+1)(N_c+2)}\mathbf{P}^{27,8}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6}$$

$$- \frac{N_c^2-N_c-2}{81N_c\,(N_c+2)}\mathbf{P}^{27,27s}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6}$$

Backup: Three gluon multiplets

SU(3) dim	1	8	10	$\overline{10}$	27	0
Multiplet	c0c0	c1c1	c11c2	c2c11	c11c11	c2c2
	$((45)^{8s}6)^{1}$	$2 \times ((45)^{8s}6)^{8s}$ or a	$((45)^{8s}6)^{10}$	$((45)^{8s}6)^{\overline{10}}$	$((45)^{8s}6)^{27}$	$((45)^{8s}6)^{0}$
	$((45)^{8a}6)^{1}$	$2 \times ((45)^{8a}6)^{8s}$ or a	$((45)^{8a}6)^{10}$	$((45)^{8a}6)^{\overline{10}}$	$((45)^{8a}6)^{27}$	$((45)^{8a}6)^{0}$
		$((45)^{10}6)^{8}$	$((45)^{10}6)^{10}$	$((45)^{\overline{10}}6)^{\overline{10}}$	$((45)^{10}6)^{27}$	$((45)^{10}6)^{0}$
		$((45)^{\overline{10}}6)^{8}$	$((45)^{10}6)^{10}$	$((45)^{\overline{10}}6)^{\overline{10}}$	$((45)^{\overline{10}}6)^{27}$	$((45)^{\overline{10}}6)^{0}$
		$((45)^{27}6)^{8}$	$((45)^{27}6)^{10}$	$((45)^{27}6)^{\overline{10}}$	$((45)^{27}6)^{27}$	$((45)^{0}6)^{0}$
		$((45)^{0}6)^{8}$	$((45)^{0}6)^{10}$	$((45)^{0}6)^{\overline{10}}$	$((45)^{27}6)^{27}$	$((45)^{0}6)^{0}$

SU(3) dim	64	35	$\overline{35}$	0
Multiplet	c111c111	c111c21	c21c111	c21c21
	$((45)^{27}6)^{64}$	$((45)^{10}6)^{35}$	$((45)^{\overline{10}}6)^{\overline{35}}$	$((45)^{10}6)^{c21c21}$
		$((45)^{27}6)^{35}$	$((45)^{27}6)^{\overline{35}}$	$((45)^{\overline{10}}6)^{c21c21}$
				$((45)^{27}6)^{c21c21}$
				$((45)^{0}6)^{c21c21}$

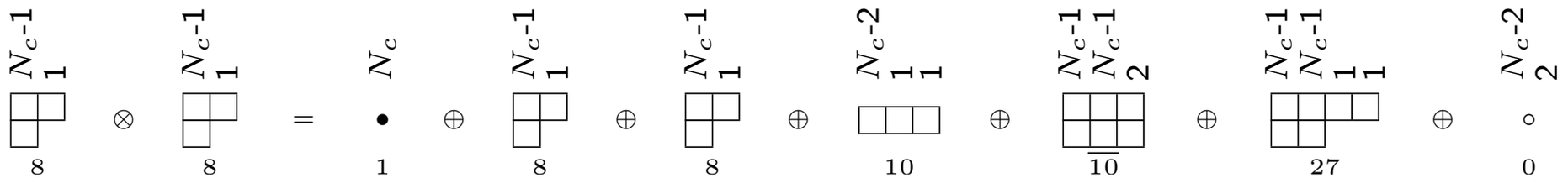
SU(3) dim	0	0	0	0	0
Multiplet	c111c3	c3c111	c21c3	c3c21	c3c3
	$((45)^{10}6)^{c111c3}$	$((45)^{\overline{10}}6)^{c3c111}$	$((45)^{10}6)^{c21c3}$	$((45)^{\overline{10}}6)^{c3c21}$	$((45)^{0}6)^{c3c3}$
			$((45)^{0}6)^{c21c3}$	$((45)^{0}6)^{c3c21}$	

Multiplets for $g_4 \otimes g_5 \otimes g_6$

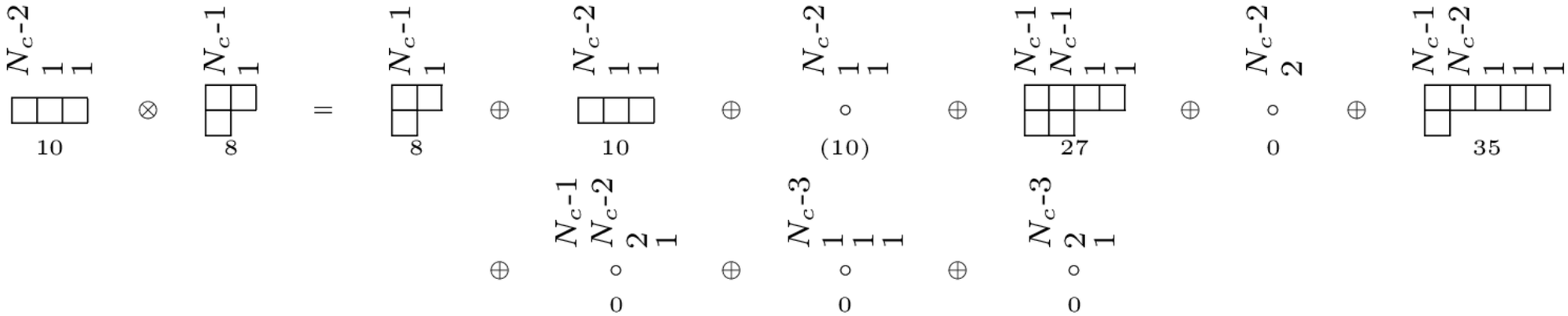
Backup: Construction of 3 gluon projectors

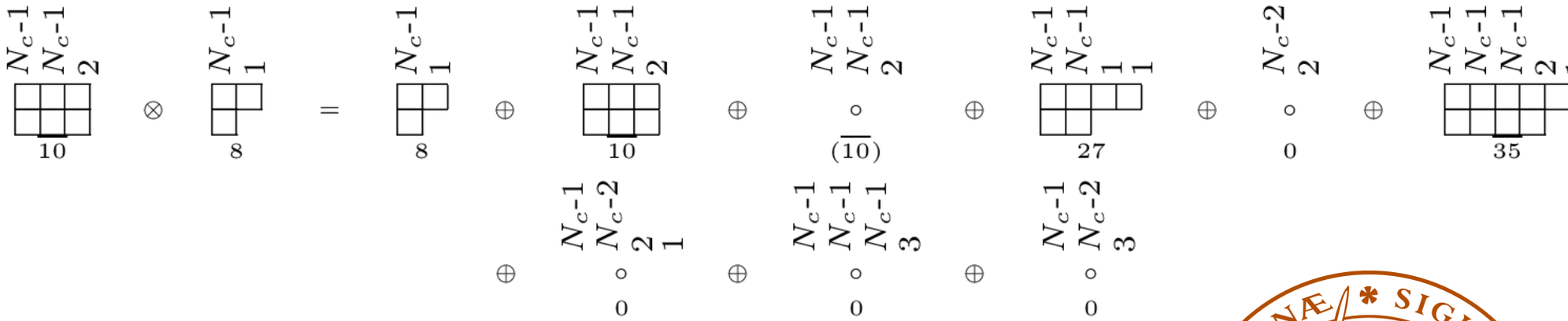
We start out by enumerating all projectors in $(8_1 \otimes 8_2) \otimes 8_3$

- Starting in a singlet, the result is trivial $1_{12} \otimes 8_3 = 8_{123}$
- If we start in an octet 8_{12}, $8_{12} \otimes 8_3$ is known from before:

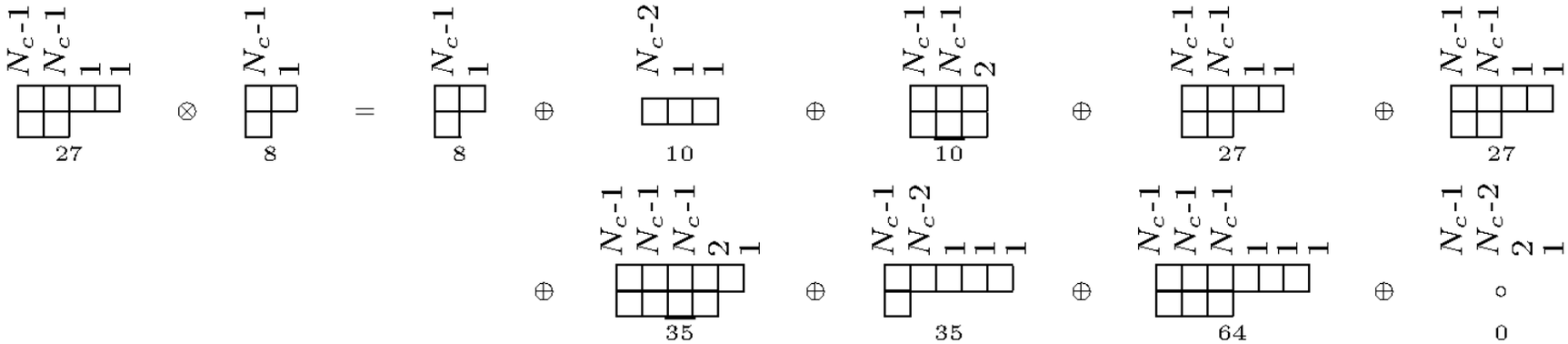


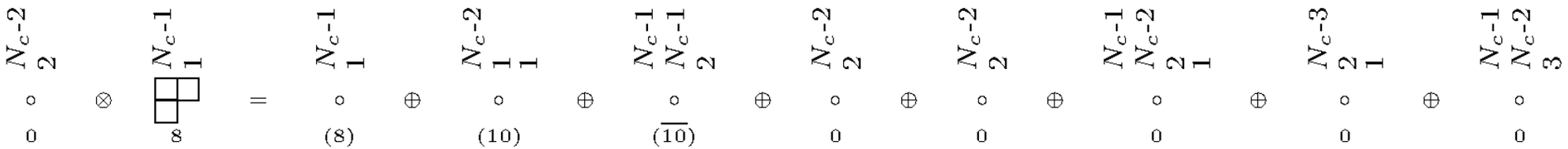
- The 3g multiplets from (anti-) decuplets





- The 3g multiplets from 27- and 0-plets

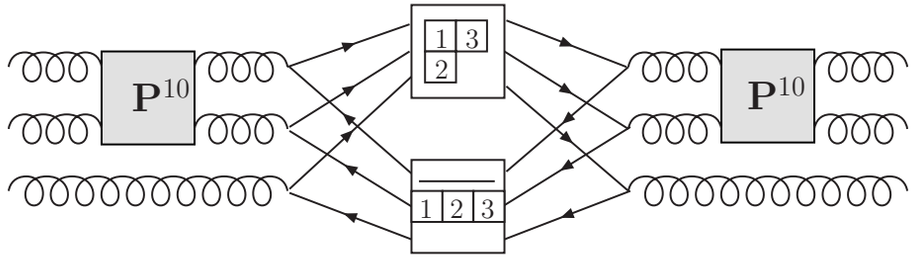




Backup: Projector construction

- Construct projectors corresponding to "old" multiplets
- Construct the tensors which will give rise to "new" projectors

$$\mathbf{T}^{10,35} \propto$$



- From these, project out "old" multiplets

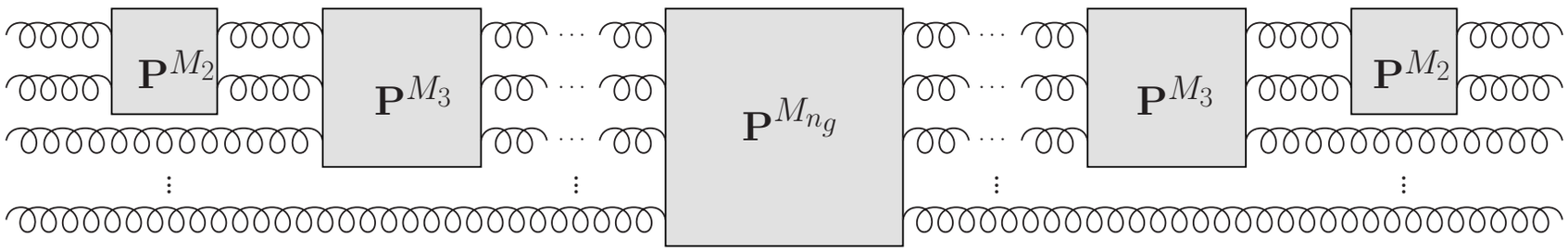
$$\mathbf{P}^{10,35} \propto \mathbf{T}^{10,35} - \sum_{m \subseteq 10 \otimes 8} \mathbf{P}^{m}\, \mathbf{T}^{10,35}$$

$\rightarrow$ "new" projectors

Backup: Projecting out "old" multiplets

This would give us a way of constructing all projectors corresponding to "new" multiplets, if we knew how to project out all old multiplets. In $g_1 \otimes g_2 \otimes g_3$, there are many 27-plets. How do we separate the various instance of the same multiplet?

- By the construction history!

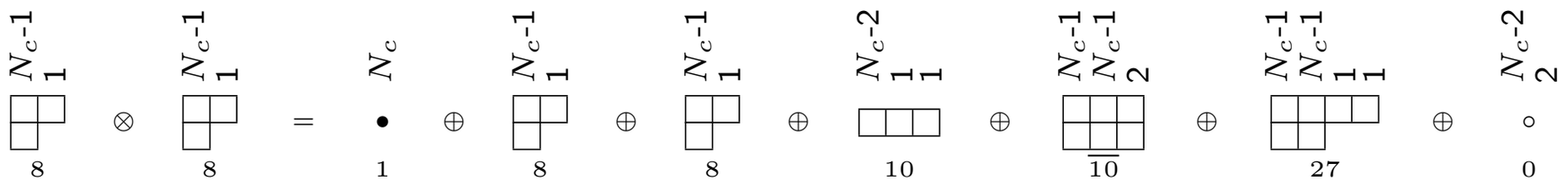


We make sure that the $n_g - \nu$ first gluons are in a given multiplet! Then the various instances are orthogonal as, at some point in the construction history, there was a different projector!

(More complicated for multiple occurrences...)

It turns out that the proof of this is really interesting:

- We find that the irreducible representations in $g^{\otimes n_g}$ for varying N_c stand in a one to one, or one to zero correspondence to each other! (For each SU(3) multiplet there is an SU(5) version, but not vice versa.)
- Every multiplet in $g^{\otimes n_g}$ can be labeled in an N_c-independent way using the lengths of the *columns*. For example



I have not seen this column notation elsewhere... have you?

Backup: Number of projection operators and basis vectors

In general, for many partons the size of the vector space is much smaller for $N_c = 3$, compared to for $N_c \to \infty$

Case	Projectors $N_c = 3$	Projectors $N_c = \infty$	Vectors $N_c = 3$	Vectors $N_c = \infty$
2g $\to$ 2g	6	7	8	9
3g $\to$ 3g	29	51	145	265
4g $\to$ 4g	166	513	3 598	14 833
5g $\to$ 5g	1 002	6 345	107 160	1 334 961

Number of projection operators and basis vectors for $N_g \to N_g$ gluons *without* imposing projection operators and vectors to appear in charge conjugation invariant combinations

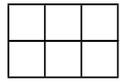
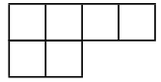
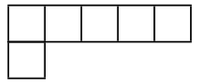
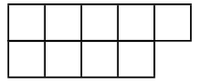
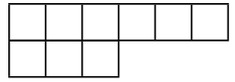
- The size of the vector spaces asymptotically grows as an exponential in the number of gluons/$q\bar{q}$-pairs for finite N_c
- For general N_c the basis size grows as a factorial

$$N_{\text{vec}}[n_q, N_g] = N_{\text{vec}}[n_q, N_g - 1](N_g - 1 + n_q) + N_{\text{vec}}[n_q, N_g - 2](N_g - 1)$$

where

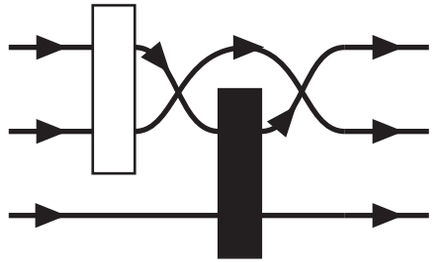
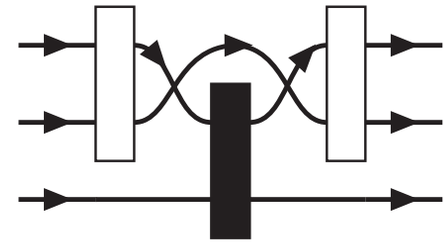
$$\begin{aligned} N_{\text{vec}}[n_q, 0] &= n_q! \\ N_{\text{vec}}[n_q, 1] &= n_q n_q! \end{aligned}$$

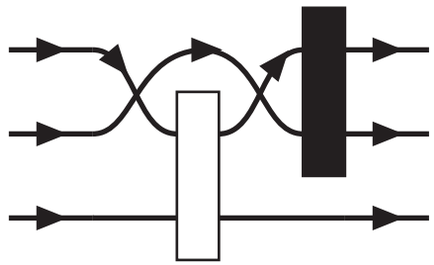
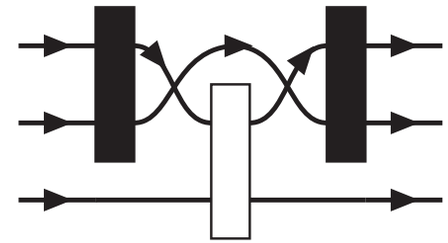
Backup: First occurrence

n_f	0	1	2	3
SU(3) Young diagrams	$\bullet =$ 		  	  

Examples of SU(3) Young diagrams sorted according to their first occurrence n_f.

Backup: The importance of Hermitian projectors

$\mathbf{P}_Y^{6,8} = \frac{4}{3}$  , $\mathbf{P}^{6,8} = \frac{4}{3}$ 

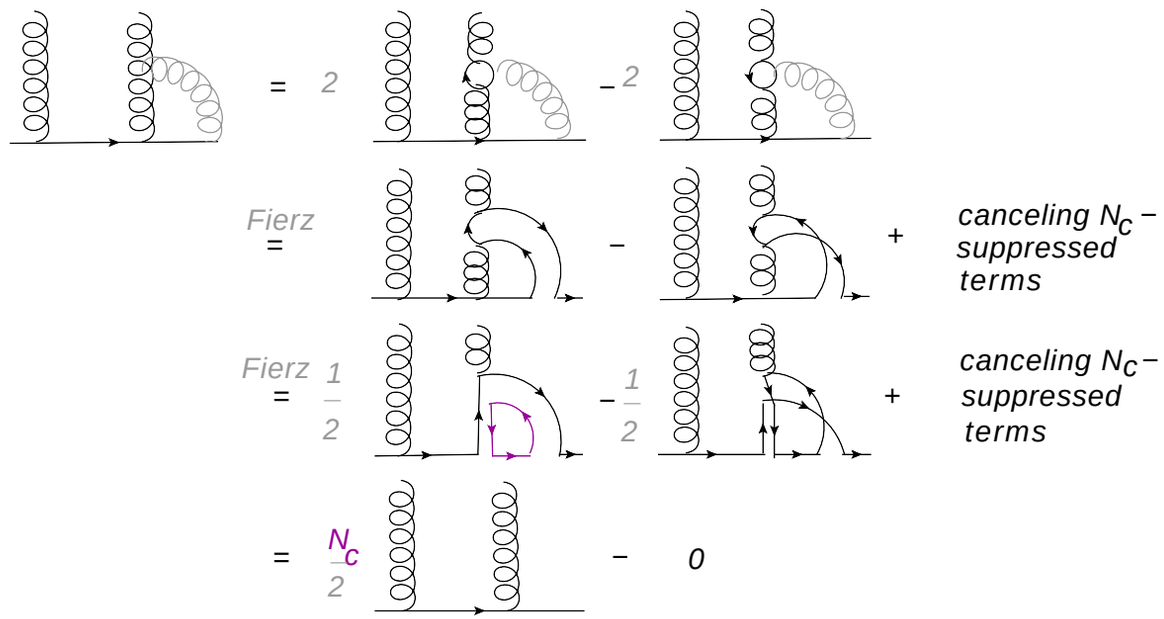
$\mathbf{P}_Y^{\bar{3},8} = \frac{4}{3}$  , $\mathbf{P}^{\bar{3},8} = \frac{4}{3}$ 

The standard Young projection operators $\mathbf{P}_Y^{6,8}$ and $\mathbf{P}_Y^{\bar{3},8}$ compared to their Hermitian versions $\mathbf{P}^{6,8}$ and $\mathbf{P}^{\bar{3},8}$.

Clearly $\mathbf{P}^{6,8\dagger}\mathbf{P}^{\bar{3},8} = \mathbf{P}^{6,8}\mathbf{P}^{\bar{3},8} = 0$. However, as can be seen from the symmetries, $\mathbf{P}_Y^{6,8\dagger}\mathbf{P}_Y^{\bar{3},8} \neq 0$.

Backup: Gluon exchange

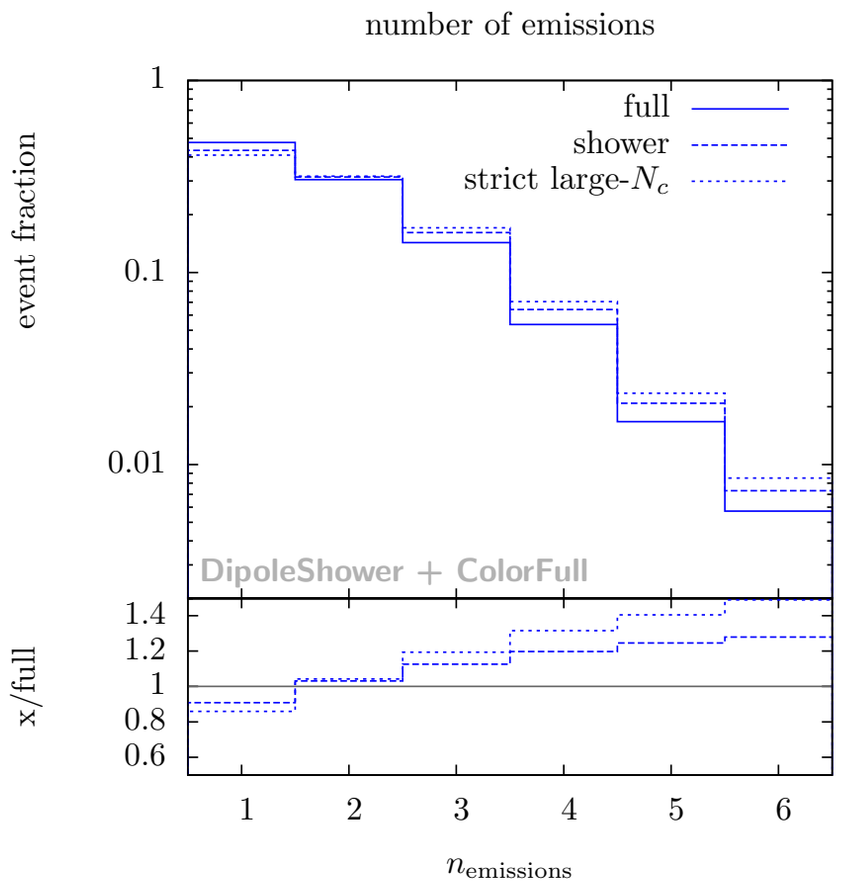
A gluon exchange in this basis "directly" i.e. without using scalar products gives back a linear combination of (at most 4) basis tensors



- N_c-enhancement possible only for near by partons
  $\rightarrow$ only "color neighbors" radiate in the $N_c \rightarrow \infty$ limit

Backup: Number of emissions

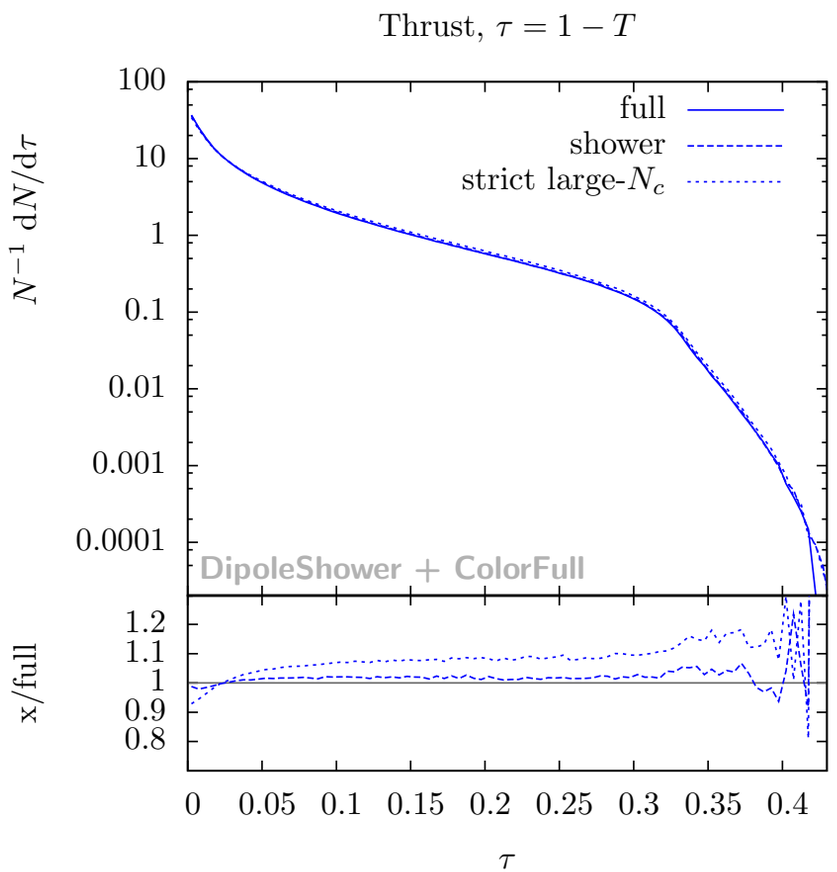
First, simply consider the number of emissions for a LEP-like setting



... this is not an observable, but it is a genuine uncertainty on the number of emissions in the perturbative part of a parton shower

Backup: Thrust

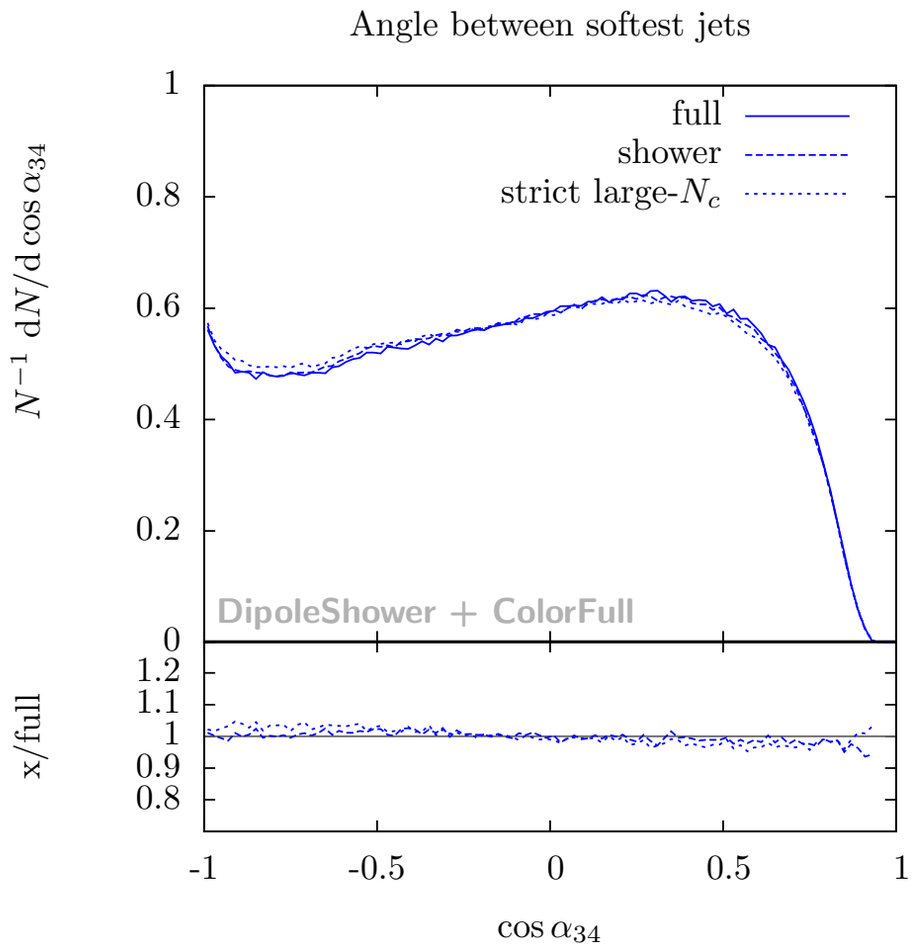
For standard observables small effects, here thrust $T = \max_{\mathbf{n}} \frac{\sum_i |\mathbf{p_i} \cdot \mathbf{n}|}{\sum_i |\mathbf{p_i}|}$



NOTE: Larger effects expected at LHC

Backup: Angular distribution

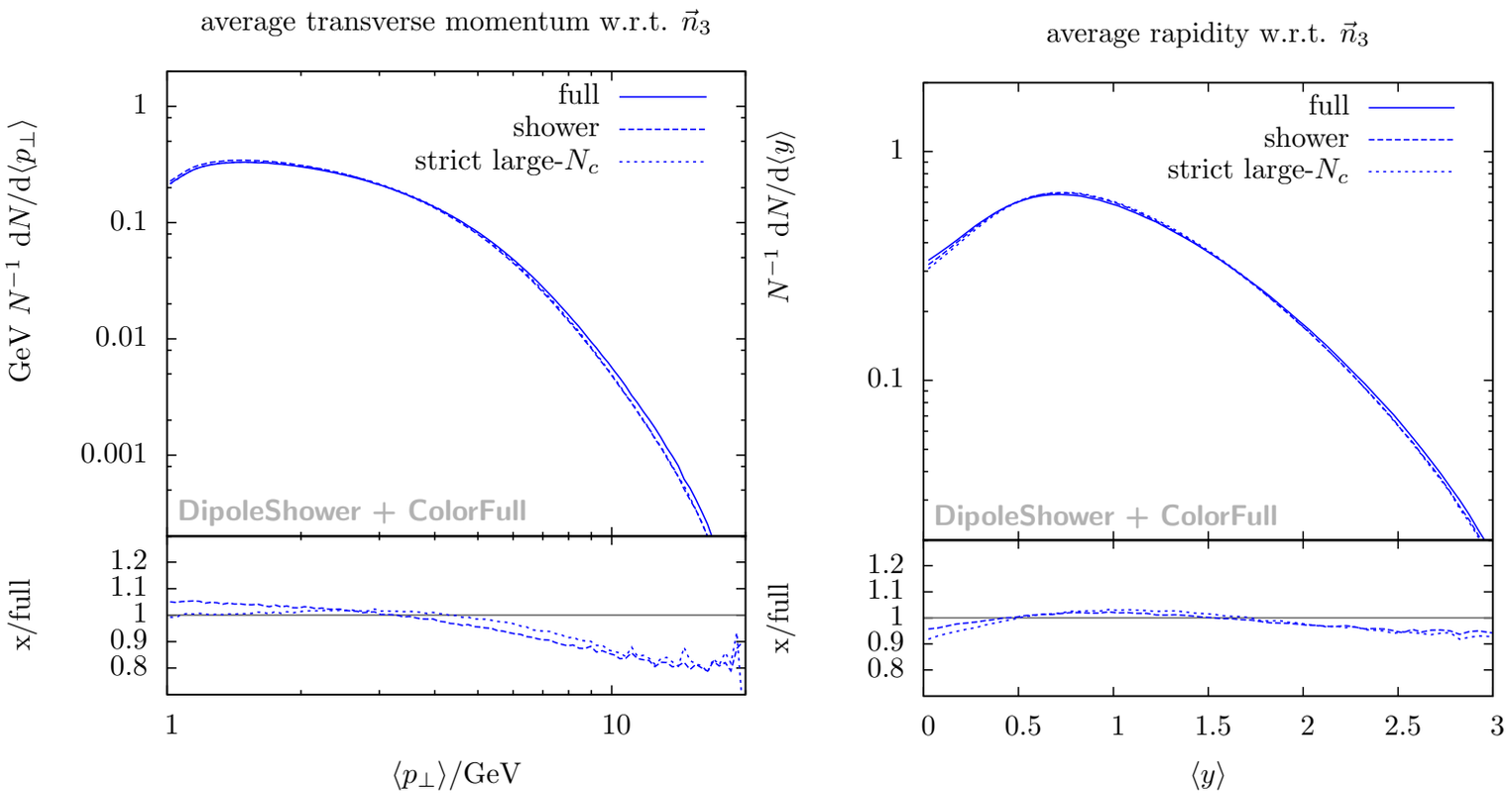
Cosine of angle between third and fourth jet



NOTE: Larger effects expected at LHC

Backup: Some tailored observables

For tailored observables we find larger differences

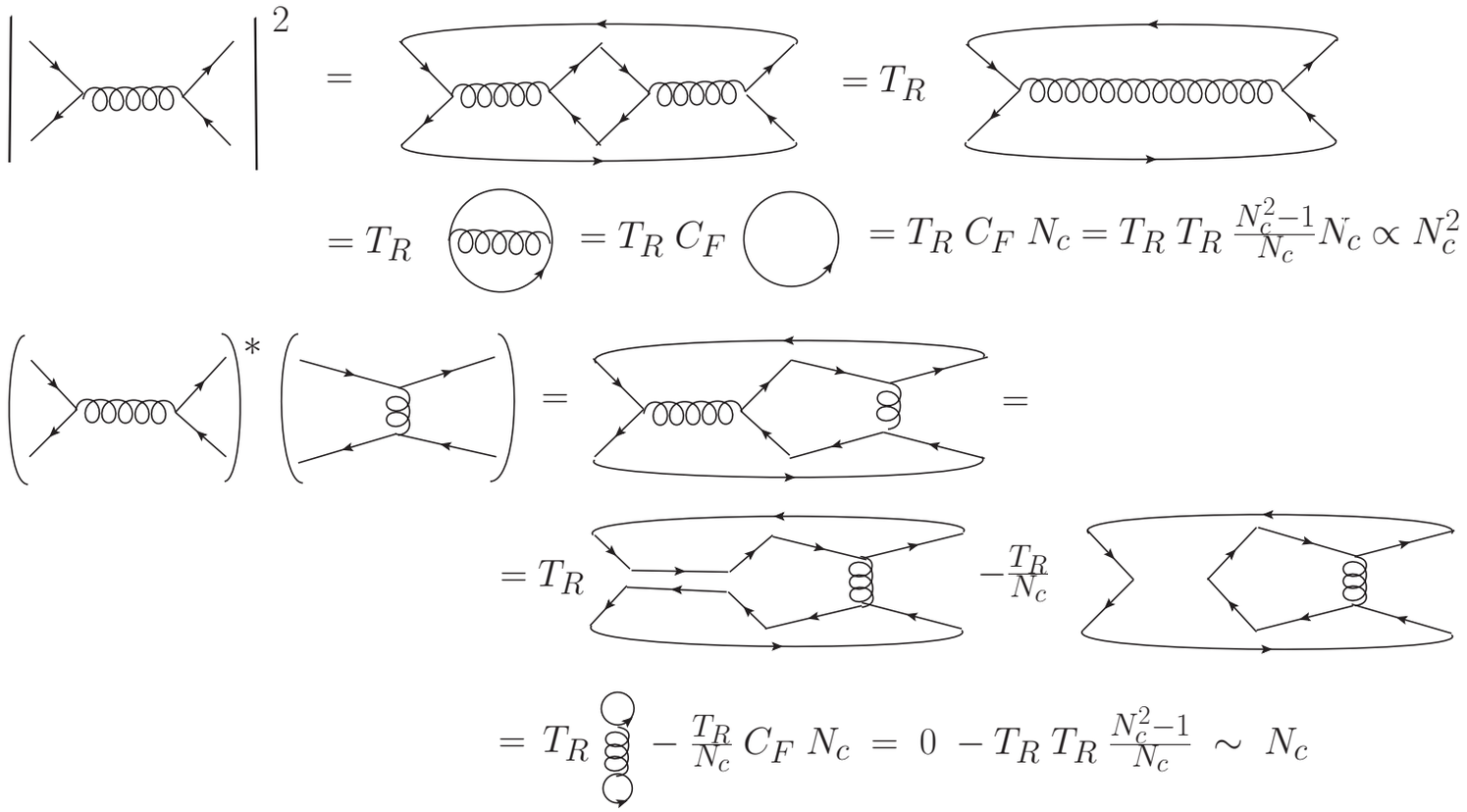


Average transverse momentum and rapidity of softer particles with respect to the thrust axis defined by the three hardest partons

NOTE: Larger effects expected at LHC

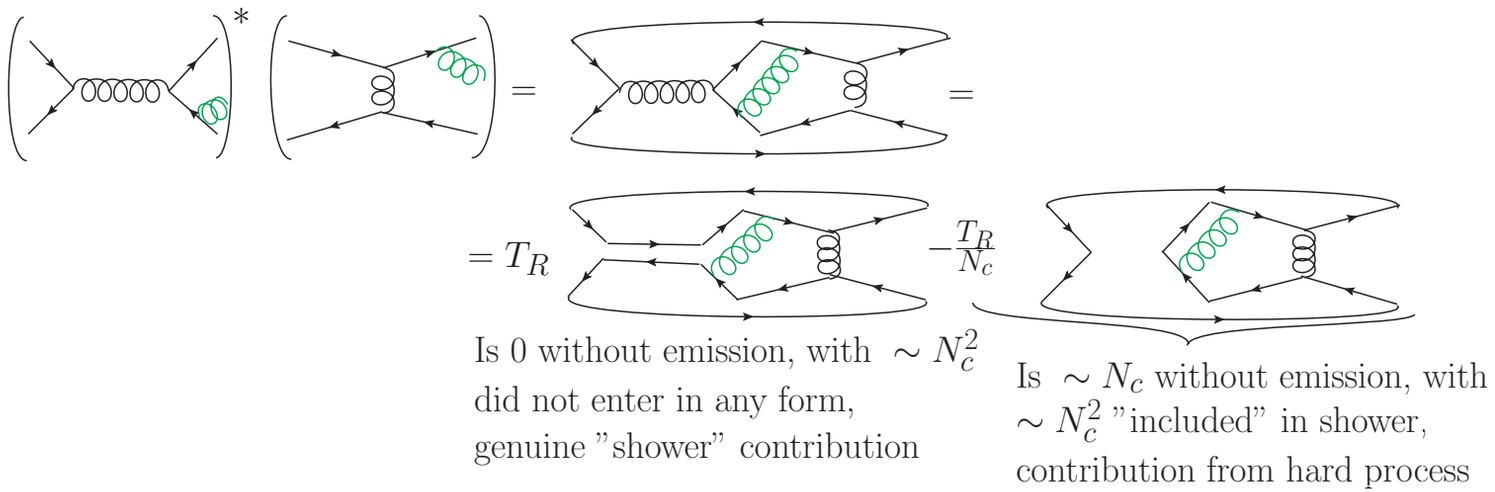
Backup: N_c-suppressed terms

That non-leading color terms are suppressed by $1/N_c^2$, is guaranteed only for same order α_s diagrams with only gluons ('t Hooft 1973)



Backup: N_c-suppressed terms

For a parton shower there may also be terms which only are suppressed by one power of N_c



The leading N_c contribution scales as N_c^2 before emission and N_c^3 after