FYTA 14 - Fluid dynamics
Supplementary exercises with astrophysical applications

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1 The extended atmosphere

In this exercise, we leave for space. But before we can do so, we have to punch through Earth's atmosphere. In order to know when we can declare ourselves as being in space, we're interested in the density structure of the atmosphere.

For simplicity, we assume an isothermal ideal atmosphere, so that $p = \rho R T_0$, where R is a "specific gas constant" $\sim 300\,\mathrm{J/(kg{\cdot}K)}$ for earth.

Note: We shall assume that quantities with a zero-index are constant, like T_0, whereas g(r) denotes a function of radius r, and p or ρ are functions as apparent from context.

a) Near the earth surface, we can assume constant gravity, $\vec{g} = -\widehat{e_r} g_0$. Use the hydrostatic balance equation $p'(r) = -g_0\rho(r)$ and the assumptions above to show that the atmosphere near the surface obeys

$$\rho(z) = \rho_0 \exp(-z/H)$$

where z is the height above ground. Also, find an expression for the "atmospheric scale height" H and give an interpretation of the introduced constant ρ_0.

b) It seems that H is related to the atmosphere's thickness. With $T_0 \sim 300\,\mathrm{K}$, $g_0 \approx 10\,\mathrm{m/s^2}$ and earth radius $r_0 \sim 7 \cdot 10^6\,\mathrm{m}$, give an order-of-magnitude estimate of H, and of H/r_0.

c) Verify that the total atmospheric mass outside a radius r_1 can be written

$$M(r > r_1) = 4\pi\rho_0 \exp\frac{r_0 - r_1}{H} \int_0^\infty \exp\left(-\frac{y}{H}\right)(r_1 + y)^2 dy$$

where $y = r - r_1$.

d) We are only interested in $r_1 \geq r_0$, and due to the exponential suppression, we realize that the integral gets its main contribution for $y \sim H$. Use your estimate of H/r_0 to motivate the approximation

$$M(r > r_1) \approx 4\pi \rho_0 r_1^2 H \exp \frac{r_0 - r_1}{H}$$

Hint: It is easiest to make approxmiations before you integrate.

e) The total atmospheric mass is clearly $M = M(r > r_0)$. Write down the fraction of mass outside r_1, $f_1 = M(r > r_1)/M$.

Your f_1 is valid for all $r_1 \geq r_0$. We will now look for the r_1 that encloses all but 1 permille of the atmospheric mass. To this end, define a relative height x_1 from $r_1 = (1+x_1)r_0$ and write f_1 as a function of x_1. Assume $x_1 \ll 1$ and make a rough estimate of the value of x_1 needed to get $f_1 \sim 10^{-3} \approx e^{-7}$. Compare the resulting height $r_1 - r_0$ to H. Then, verify consistency of the assumption: the r_1 you found should indeed satisfy $\frac{r_1}{r_0} - 1 \ll 1$.

f) A very simple mass estimate is to write earth surface pressure p_0 as a force F divided by earth area $4\pi r_0^2$, and approximate $F \sim M g_0$. Using the isothermal ideal model, find an expression for M this way, and compare to $M(r > r_0)$ given in (e).

g) We now acknowledge that gravity actually changes with distance from the center of the gravitating body, $g(r) = -\widehat{e}_r \frac{GM_{\text{earth}}}{r^2} = -\widehat{e}_r g_0 \frac{r_0^2}{r^2}$. The equation for hydrostatic equilibrium now reads $p'(r) = -\rho(r) g(r)$. With otherwise the same assumptions as before, and the same interpretation of ρ_0, show that the atmosphere obeys

$$\rho(r) = \rho_0 \exp\left[r_B \left(\frac{1}{r} - \frac{1}{r_0} \right) \right]$$

and find an expression for the "Bondi radius" r_B.

h) Taylor expand the corrected gravity function for small relative heights x, defined by $r = (1+x)r_0$, and make a comment if the corrections are important within the 0.999-distance estimated above.

i) At large distances, the atmosphere gets extremely dilute, and we may worry about the huge "test volumes" needed to apply the continuum approximation. Calculate the asymptotic density $\rho(r \to \infty)$. Use that to discuss the lower limit of the total atmospheric mass

$$M = 4\pi \int_{r_0}^{\infty} \rho(r) r^2 dr.$$

Do you think we can rely on the continuum approximation for the entire atmosphere? (Comment: the well-being of the continuum approxmiation is not the only concern at large distances – radiative effects also start to play a role.)