Linearized Waves

Summary of wave equation calculations in FYTA14, Fluid Dynamics.

1 Surface Gravity Waves

The book discusses these assuming irrotational flow and working with the velocity potential (p.424-428). That is not included in this course, so here comes an alternative derivation of water wave equations.

1.1 Small amplitude waves

We let $z = 0$ be at the average surface level and $z = -d$ at the (assumed flat) bottom. We consider a plane wave with amplitude a, wavelength λ and angular frequancy ω, so that the surface is described by $z_{\text{surface}} = h(x, t)$, where

$$h(x,t) = a\cos(kx - \omega t), \tag{1}$$

where $k = 2\pi/\lambda$ is the wave number.

We consider *small amplitude waves*, meaning that a is the smallest lenght-scale in the system: $a \ll \lambda$ and $a \ll d$.

We also consider small velocities, meaning $|(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}| \ll |\frac{\partial}{\partial t}\boldsymbol{v}|$, so that the Euler equation can be linearized. We will evantually discover that this does not restrict us much: reasonable conditions relate this linearization to $a \ll \lambda$.

1.2 Equations and boundary conditions

Once the advective term is discarded, the "generalized potential",

$$\Phi^* = g_0 z + \frac{1}{\rho_0}p, \tag{2}$$

solves the Laplace equation for incompressible water:

$$\nabla^2\Phi^* = \nabla\cdot(\nabla\Phi^*) = -\nabla\cdot\frac{\partial\boldsymbol{v}}{\partial t} = -\frac{\partial}{\partial t}(\nabla\cdot\boldsymbol{v}) = 0. \tag{3}$$

Many boundary conditions restrict v_z, and it is convenient to work with the function

$$\zeta = \frac{\partial v_z}{\partial t} = -\frac{\partial\Phi^*}{\partial z}. \tag{4}$$

The function ζ also solves the Laplace equation ($\nabla^2\frac{\partial\Phi^*}{\partial z} = \frac{\partial}{\partial z}\nabla^2\Phi^* = 0$) and satisfies the boundary conditions

$$\zeta(z=-d) = 0 \tag{5}$$

$$\zeta(z=h) = \frac{\partial^2 h}{\partial t^2} = -\omega^2 a\cos(kx - \omega t) \tag{6}$$

The first condition just states that the vertical velocity must be zero at the bottom. Since this must hold at all times, we also get $\frac{\partial}{\partial t}v_z = 0$ at the bottom.

The second condition describes the surface, and is actually an approximation relying on $a \ll \lambda$. It can be derived like this: Consider a co-moving particle at the surface. At time t it is at horizontal position x and height $h(x,t)$. A time Δt later, it has moved to horizontal position $x + v_x \Delta t$, and therefore to the height $h(x + v_x \Delta t, t + \Delta t)$. During the same time, it has moved the vertical distance $v_z \Delta t$, so that $h(x + v_x \Delta t, t + \Delta t) - h(x,t) = v_z \Delta t$. Dividing by Δt and taking the limit $\Delta t \to 0$ gives

$$v_z = \frac{D}{Dt} h = \frac{\partial h}{\partial t} + v_x \frac{\partial h}{\partial x}. \tag{7}$$

It may feel risky to apply the co-moving derivative to h, which is not really a field, but a boundary, but our detailed argument above provides motivation. Finally, we note that $|\frac{\partial}{\partial x} h| \leq ka \ll 1$ when $a \ll \lambda$. Thus, we get $v_z = \frac{D}{Dt} h \approx \frac{\partial}{\partial t} h$ and eq. (6).

1.3 Solution

There are many representations of the general solution to the Laplace equation. Here, it is convenient separate the x and z variables. We assign $\zeta(x,z,t) = X(x,t)Z(z,t)$. Suppressing the time dependence in our notation for a while, we get

$$0 = \nabla^2 \zeta = \frac{\partial^2}{\partial x^2}(XZ) + \frac{\partial^2}{\partial z^2}(XZ) = X''(x)Z(z) + X(x)Z''(z), \tag{8}$$

which can be re-arranged to

$$\frac{X''(x)}{X(x)} = -\frac{Z''(z)}{Z(z)}. \tag{9}$$

The LHS is independent of z and the RHS is indpendent of x, so both sides must be constant in space. In order to reduce the number of symbols, we now anticipate an important result, and write the constant $-k^2$. It is a reasonable guess: we get oscillations in the x-direction

$$X''(x) = -k^2 X(x) \Longrightarrow X \propto \exp(\pm ikx) \tag{10}$$

in agreement with $h(x,t)$. We then get exponential changes in the z-direction

$$Z''(z) = +k^2 Z(z) \Longrightarrow Z \propto \exp(\pm kz). \tag{11}$$

With two independent solutions each for X and Z, the general solution becomes a linear combination with four unknown constants C_i,

$$\zeta = C_1 \exp(kz + ikx) + C_2 \exp(kz - ikx) + C_3 \exp(-kz + ikx) + C_4 \exp(-kz - ikx). \tag{12}$$

This is just one of many ways to represent the general solution. We may for example re-define undetermined constants to rewrite[1]

$$C_1 \exp(kz) + C_3 \exp(-kz) = D_1 \sinh(kz) + D_3 \cosh(kz) = A_1 \cos(kz + a_1) = B_1 \sinh(kz + b_1). \tag{13}$$

The final represenation, involving $\sinh(kz + b_1)$ is particularly useful in this problem, since the boundary condition $\zeta(z = -d) = 0$ immediately gives $b_1 = kd$. The same exercise on the C_2 and C_4 terms allows our solution to be written

$$\zeta = B_1 \sinh(kz + kd) \exp(ikx) + B_2 \sinh(kz + kd) \exp(-ikx). \tag{14}$$

[1] If you are new to these manipulations, it can be instructive to derive relations between the constants. It is easiest to express the C:s in terms of other constants. The reverse relations are a bit uglier. Note that since our current constants are still undetermined, we do not really need to know the relations: whatever pair we select will be determined directly by boundary conditions.

We are left with two undetermined constants, and again we pick wisely between the formulations

$$B_1 \exp(ikx) + B_2 \exp(-ikx) = A_0 \cos(kx) + B_0 \sin(kx) = A \cos(kx + \theta) = B \sin(kx + \phi). \tag{15}$$

The version involving constants A and θ is most similar to the $z = h$ boundary condition,

$$-\omega^2 a \cos(kx - \omega t) = \zeta(z = h) = A \sinh(kh + kd) \cos(kx + \theta). \tag{16}$$

We find $\theta = -\omega t$, and remember that our "constants" were just independent of x and z, so that time dependence is allowed.

We also see that we have correctly introduced k above. (Formally, we should have introduced a new symbol $-K^2(t)$, and allowed a sum over several K, or even an integral over a K-density $f(K)$. At this point in the calculations, we would then note that only a real $K = k$ can fit the surface boundary condition at hand.)

Finally, we seem to get

$$A \stackrel{?}{=} -\frac{\omega^2 a}{\sinh(kh + kd)}, \tag{17}$$

but then A gains a forbidden space dependence, since $h = h(x, t)$. This may surprise those who have learnt that variable separation is a fool-proof way to find solutions to linear differential equations. However, while the Laplace equation is indeed linear, our boundary conditions are not: they include $h(x, t)$ which depends on the solution $\boldsymbol{v}$.

To be able to continue, we note that $|kh| \leq ka \ll 1$ gives $\exp(\pm kh) \approx 1 \Rightarrow \sinh(kh + kd) \approx \sinh(kd)$. Within the approximations already made, we therefore find

$$A = -\frac{\omega^2 a}{\sinh(kd)}. \tag{18}$$

(So, A turns out not to have any time-dependence.) We have the solution for ζ:

$$\zeta = \frac{\partial v_z}{\partial t} = -\frac{\partial}{\partial z} \Phi^* = -\omega^2 a \frac{\sinh(kz + kd)}{\sinh(kd)} \cos(kx - \omega t). \tag{19}$$

1.4 Pressure boundary condition

Integrating eq. (19) over z gives

$$\Phi^* = \frac{\omega^2 a}{k} \frac{\cosh(kz + kd)}{\sinh(kd)} \cos(kx - \omega t) + B(x, t). \tag{20}$$

To get control over the unknown function $B(x, t)$, we look at pressure. Constant air pressure p_0 at the surface gives $\Phi^*(z = h) = g_0 h + p_0/\rho_0$. Again using $\exp(\pm kh) \approx 1$, so that $\Phi^*(z = h) \approx \Phi^*(z = 0)$, we get

$$g_0 h(x, t) + \frac{p_0}{\rho_0} \approx \Phi^*(z = 0) = \frac{\omega^2}{k \tanh(kd)} h(x, t) + B(x, t) \tag{21}$$

To our relief, the unknown B is just a constant, p_0/ρ_0.

1.5 Velocities

With $B = p_0/\rho_o$, we get

$$\frac{\partial v_x}{\partial t} = -\frac{\partial}{\partial x} \Phi^* = \omega^2 a \frac{\cosh(kz + kd)}{\sinh(kd)} \sin(kx - \omega t) \tag{22}$$

We integrate this and eq. (19) over t to get the velocity up to an unknown steady component $\boldsymbol{v}_0(x,z)$:

$$v_x = \omega a \frac{\cosh(kz+kd)}{\sinh(kd)} \cos(kx-\omega t) + v_{0,x}(x,z), \tag{23}$$

$$v_z = \omega a \frac{\sinh(kz+kd)}{\sinh(kd)} \sin(kx-\omega t) + v_{0,z}(x,z). \tag{24}$$

From $\frac{\partial}{\partial z}\cosh(kz+kd) = k\sinh(kz+kd)$ and $\frac{\partial}{\partial x}\sin(kx-\omega t) = k\cos(kx-\omega t)$ we find the curl

$$\nabla \times \boldsymbol{v} = \nabla \times \boldsymbol{v}_0. \tag{25}$$

(Indeed, already from $\frac{\partial}{\partial t}\boldsymbol{v} = -\nabla\Phi^*$ we get time-independent curl: $\frac{\partial}{\partial t}\nabla\times\boldsymbol{v} = \nabla\times\frac{\partial}{\partial t}\boldsymbol{v} = -\nabla\times\nabla\Phi^* = 0$.)

Since $\nabla\times\boldsymbol{v}_0 \neq 0$ does not destroy any arguments in our derivation, the solutions we have found for the time-dependent part do not require the assumption of irrotational flow. However, our assumption $|(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}| \ll |\frac{\partial}{\partial t}\boldsymbol{v}|$ requires $\boldsymbol{v}_0$ to be small: since the steady $\boldsymbol{v}_0$ does not contribute to $\frac{\partial}{\partial t}\boldsymbol{v}$, we cannot allow terms like $(\boldsymbol{v}_0\cdot\nabla)\boldsymbol{v}$ to be large. Instead of searching fot the most general constraints on $\boldsymbol{v}_0$, we restrict ourselves to the often applicable situation of constant $\boldsymbol{v}_0$. By simply transforming to the coordinate system where $\boldsymbol{v}_0 = 0$, we get $|\boldsymbol{v}| \sim \omega a$, $|\frac{\partial}{\partial t}\boldsymbol{v}| \sim \omega^2 a$ and $|\nabla\boldsymbol{v}| \sim k\omega a$, so that

$$\frac{|(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}|}{|\frac{\partial}{\partial t}\boldsymbol{v}|} \sim \frac{k\omega^2 a^2}{\omega^2 a} = ka \ll 1. \tag{26}$$

This justifies the early linearization of the equations for small-amplitude waves.

1.6 Dispersion relation

When we watch a wave, we cannot really observe $\boldsymbol{v}$, which describes the velocity of the water flow. (Perhaps a floating, co-moving object can give us a hint of $\boldsymbol{v}$ in its particual point.) What we do observe is the phase velocity $c = \omega/k$, which describes how wave crests move, and possibly the group velocity $c_g = \frac{\partial\omega}{\partial k}$, which describes how the whole wave package travels (for example, how the waves move away from a speeding boat and eventually reach the shore).

These velocities are determined by a *dispersion relation*, $\omega = \omega(k)$. In our case, we find the dispersion relation in eq. (21). It not only determines that B is constant, it also leads to

$$\omega^2 = g_0 k \tanh(kd). \tag{27}$$

This is perhaps not so pretty, but we will have a look at the phase and group velocities in two relevant limits.

1.7 Two limits

Small amplitude waves must satisfy $a \ll d$ and $a \ll \lambda$, but impose no other demands on depth d and wavelength λ. Many practical situations will be represented by either the *shallow water limit*, $a \ll d \ll \lambda$, or the *deep water limit*, $a \ll \lambda \ll d$. Note that it is just the relative sizes of d and λ that determines the limit. A tsunami in the middle of a 4 km deep ocean is actually a shallow water wave, since the wavelength can be roughly 100 km.

In the shallow water limit, we simplify the dispersion relation using $\tanh(kd) \approx kd$, which is valid for $kd \ll 1$. We get $v = ck$ with $c = \sqrt{g_0 d}$. Since c is a constant, this is a *non-dispersive* wave, where group velocity and phase velocities are equal, $c_g = \frac{\partial\omega}{\partial k} = c$.

Note that c is constant with respect to k, but depends on depth d. A problem in the book asks you to relate this to the observation that waves reaching a beach typically are parallell to the shore.

The deep water limit, $kd \gg 1$, gives $\cosh(kd+\alpha) \approx \frac{1}{2}\exp(kd+\alpha)$ and $\sinh(kd+\alpha) \approx \frac{1}{2}\exp(kd+\alpha)$. (In our equations, we have $\alpha = 0$ or $\alpha = kz$.) The solutions then become independent of d:

$$v_x = \omega a \exp(kz)\cos(kx - \omega t), \tag{28}$$
$$v_z = \omega a \exp(kz)\sin(kx - \omega t). \tag{29}$$

This is reasonable: when the bottom is really far away, it does not matter for the surface wave how far "really far" is.

In this case, the dispersion relation simplifies, using $\tanh(kd) \approx 1$, into $\omega^2 = g_0 k$. The phase velocity (or "celerity") becomes $c = \frac{\omega}{k} = \sqrt{g_0/k}$ and the group velocity becomes $c_g = \frac{\partial\omega}{\partial k} = c/2$. The fact that $c_g < c$ is something you can see behind a speeding boat: the wave crests extend in a different angle than the wave package itself. The book makes an additional, beautiful observation about waves of different wavelengths reaching the shore at different times during the progression of a storm.

2 Damped Sound Waves

The book discusses these making early simplifications, and working with trigonometric functions rather than complex exponentials. Experience is that some of the arguments can be hard to follow, in particular the effects of the limit $\omega \ll \omega_0$, so here comes an alternative approach.

Since the viscosity terms involving $\nabla^2 \boldsymbol{v}$ and $\nabla(\nabla\cdot\boldsymbol{v})$ are already linear in $\boldsymbol{v}$, we just attach them to the linearized equations we had before, to get (with zero gravity and constant equilibrium pressure p_0)

$$\frac{\partial}{\partial t}\Delta\rho + \rho_0 \nabla\cdot\boldsymbol{v} = 0 \tag{30}$$
$$\frac{\partial}{\partial t}\boldsymbol{v} = -\frac{c_0^2}{\rho_0}\nabla\,\Delta\rho + \frac{1}{\rho_0}\eta\nabla^2\boldsymbol{v} + \frac{1}{\rho_0}(\zeta + \frac{1}{3}\eta)\nabla(\nabla\cdot\boldsymbol{v}). \tag{31}$$

As before we have introduced the sound velocity

$$c_0^2 = p'(\rho_0). \tag{32}$$

Using $\nabla\cdot[\nabla(\nabla\cdot\boldsymbol{v})] = \nabla^2(\nabla\cdot\boldsymbol{v})$ and $\nabla\cdot(\nabla^2\boldsymbol{v}) = \nabla^2(\nabla\cdot\boldsymbol{v})$ we then get

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\Delta\rho &= -\rho_0\frac{\partial}{\partial t}\nabla\cdot\boldsymbol{v} = -\rho_0\nabla\cdot(\frac{\partial}{\partial t}\boldsymbol{v}) = \\ &= c_0^2\nabla^2\Delta\rho - (\zeta + \frac{4}{3}\eta)\nabla^2\nabla\cdot\boldsymbol{v} = \\ &= c_0^2\nabla^2\Delta\rho + \frac{1}{\rho_0}(\zeta + \frac{4}{3}\eta)\nabla^2\frac{\partial}{\partial t}\Delta\rho = \\ &= c_0^2\nabla^2\Delta\rho + \frac{c_0^2}{\omega_0}\nabla^2\frac{\partial}{\partial t}\Delta\rho,\end{aligned} \tag{33}$$

where the last step defines

$$\omega_0 = \frac{c_0^2\rho_0}{\zeta + \frac{4}{3}\eta}. \tag{34}$$

Trying variable separation,

$$\rho(\boldsymbol{r}, t) = T(t)S(\boldsymbol{r}), \tag{35}$$

gives

$$\frac{T''(t)}{T(t)+\frac{1}{\omega_0}T'(t)} = c_0^2\frac{\nabla^2 S(\boldsymbol{r})}{S(\boldsymbol{r})}. \tag{36}$$

The LHS is independent of space, and the RHS is independent of time, so both must be constant.

To decide how to express that constant, we restrict our discussion to sound waves with preserved amplitude over time. This corresponds to a situation where a source continues to generate sound, so that the amplitude in every point is preserved, but where dampening influences how the amplitude changes with distance from the source. To avoid any amplitude changes with time, we therefore assign two independent solutions

$$T(t) = \exp(\pm i\omega t), \tag{37}$$

where ω is real and (by convention) positive. Clearly, ω is the angular frequency of the sound generated by the source.

Once it is time to fit to boundary conditions, it is probably better to use trigonometric functions $\cos(\omega t)$ and $\sin(\omega t)$, but for a while longer, our work will be easier if we use complex exponentials.

As in the book, we now restrict the discussion to a plane wave propagating in the x direction, so that $S(\boldsymbol{r}) = S(x)$. The space dependence is then determined by

$$c_0^2\frac{S''(x)}{S(x)} = -\frac{\omega^2}{1\pm i\frac{\omega}{\omega_0}} = -\frac{\omega^2}{1+\frac{\omega^2}{\omega_0^2}}\left(1\mp i\frac{\omega}{\omega_0}\right). \tag{38}$$

(Household tip: it is always a good idea to make denominators real.)

This is an excellent moment to look at the simplifying limit $\omega \ll \omega_0$. From

$$1+i\varepsilon = (1+i\frac{1}{2}\varepsilon)^2 + \mathcal{O}(\varepsilon^2) \tag{39}$$

we get

$$\frac{S''(x)}{S(x)} \approx -\frac{\omega^2}{c_0^2}\left(1\mp i\frac{\omega}{2\omega_0}\right)^2 = -\left(k\mp i\kappa\right)^2. \tag{40}$$

where the last step defines

$$k = \frac{\omega}{c_0}, \tag{41}$$

$$\kappa = \frac{\omega^2}{2c_0\omega_0}. \tag{42}$$

We have introduced two possible solutions, related to a sign choice in eq. (37), and now a second sign choice is coming. To keep track of related signs, we introduce $s_1 = \pm 1$, and write the options we have so far considered

$$T(t) = \exp(s_1 i\omega t), \tag{43}$$

$$S''(x) = -\left(k - s_1 i\kappa\right)^2 S(x). \tag{44}$$

For solutions to S, we introduce another sign variable $s_2 = \pm 1$ and write

$$S(x) \propto \exp[s_2 i(k - s_1 i\kappa)x] = \exp(s_2 ikx + s_1 s_2\kappa x). \tag{45}$$

The general solution is a linear combination of the four possible functions

$$[\Delta\rho]_{s_1,s_2} = \exp(s_1 s_2\kappa x)\exp[i(s_2 kx + s_1\omega t)]. \tag{46}$$

For waves propagating in the positive x direction, the oscillating factor must be a function of $kx - \omega t$, which means $s_1 = -s_2$. For such waves, the remaining two independent solutions are

$$[\Delta\rho]_{s_2} \propto \exp(-\kappa x)\exp[s_2 i(kx - \omega t)]. \tag{47}$$

The factor $\exp(-\kappa x)$ describes dampening of the amplitude for larger x, as is expected for a wave propagating in the positive x direction and suffering energy loss due to viscosity. From $\kappa/k = \omega/2\omega_0$ we see that $\omega \ll \omega_0 \Longrightarrow \kappa \ll k$. The penetration depth $1/\kappa$ is therefore large compared to the wavelength $\lambda = 2\pi/k$.

We finally return to the real world (pun unavoidable). To relate to the book, we look at the options to write the general solution as in eq. (15), and pick

$$\Delta\rho = \rho_1 \exp(-\kappa x)\sin(kx - \omega t - \theta), \tag{48}$$

with two constants ρ_1 and θ, to be determined by boundary condtions. We note that our initial linearization assumptions require $\rho_1 \ll \rho_0$.

3 Damped Shear Water Wave

A floor at $z = 0$ moves back and forth in the x direction with velocity $U\cos(\omega t)$. The water above will follow the wave motion, but viscosity will dampen the amplitudes far away.

We look for a solution $\boldsymbol{v} = (v_x(z,t), 0, 0)$. We get $(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = 0$ and $\nabla\cdot\boldsymbol{v} = 0$, and the zero gravity Navier–Stokes equation reduces to

$$\frac{\partial}{\partial t} v_x = \nu \frac{\partial^2}{\partial z^2} v_x. \tag{49}$$

We try variable separation $v_x(z,t) = T(t)u(z)$ and find

$$\frac{T'(t)}{T(t)} = \nu \frac{u''(z)}{u(z)}. \tag{50}$$

We consider a situation where the moving floor maintains the velocity amplitude over time, and assign

$$T(t) = \exp(\pm i\omega t) = \exp(s_1 i\omega t). \tag{51}$$

Here, we anticipate the constant in front of t to be ω, to fit the boundary condition $v_x = U\cos(\omega t)$ at $z = 0$. We also introduce a sign variable $s_1 = \pm 1$. From $u''/u = s_1 i\omega/\nu$ we find

$$u = \exp(s_2 a z), \tag{52}$$

where $s_2 = \pm 1$ and

$$a^2 \quad = \quad s_1 i \frac{\omega}{\nu} = \frac{\omega}{2\nu}(i + s_1)^2. \tag{53}$$

The final step is best verified by going backwards and noting that $i^2 + s_1^2 = -1 + 1 = 0$. With $k = \sqrt{\omega/2\nu}$ we get $a = (i + s_1)k$ and the general solution is a linear combination of

$$[v_x]_{s_1,s_2} = \exp(s_2 s_1 k z)\exp[i(s_2 k z + s_1 \omega t)]. \tag{54}$$

Looking for solutions above the floor, with velocities vanishing at $z \to \infty$ we select $s_2 s_1 = -1$. Looking at the options in eq. (15), we decide to write the general linear combination

$$v_x = v_0 \exp(-kz)\cos(kz - \omega t + \theta) = U\exp(-kz)\cos(kz - \omega t), \tag{55}$$

with two undetermined constants v_0 and θ immediately determined by the motion of the floor at $z = 0$. The shear wave propagates in the positive z direction, as it should, considering the setup with our floor.

We note that the penetration depth $1/k$ is comparable to the wavelength $2\pi/k$. One single wavelength away from the floor, at $z = \lambda$, the amplitude is reduced by a factor $\exp(-2\pi) \sim 10^{-6\pi/7} \sim 0.001$. Note that this result is independent of ν. As the book concludes, this is not much of a wave...