Exercises FYTA14, 2018. Some answers.

Appendix C

1. *Basic vector calculus operations*

a)

$$\begin{aligned} \nabla\phi &= (2xy,\, x^2,\, 1) \\ \nabla\phi &= (\cos x \cos y,\, -\sin x \sin y,\, 0) \end{aligned}$$

b)

$$\begin{aligned} \nabla \cdot \mathbf{u} &= 3x \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned}$$

c)

$$\begin{aligned} \nabla \times \mathbf{u} &= (2y - 3,\, 0,\, y) \\ \nabla \times \mathbf{u} &= (z \cos x \sin y,\, -z \sin x \cos y,\, \sin x \sin y + 2x \cos x \sin y) \end{aligned}$$

d) Show that.

e) Show that.

f) $\nabla^2 \mathbf{u} = (2,\, 0,\, 2)$, $\nabla(\nabla \cdot \mathbf{u}) = (3,\, 0,\, 0)$.

2. *Book C.1-C.3*

Show that.

3. *Book C.4*

C.4b: We must keep the parenthesis in $\nabla \cdot (\nabla \times \boldsymbol{V})$ and $\boldsymbol{V} \times (\nabla \times \boldsymbol{V})$. It is also safer to keep the parenthesis in $\nabla(\nabla \cdot V)$ and $(\nabla \cdot \nabla)\boldsymbol{V}$, though the book notation allows them to be omitted.

4. *Field Potentials*

$$\begin{aligned} &a) & \Phi &= xy + C, \\ &b) & \nabla \times \mathbf{F} &\neq (0,0,0), \\ &c) & \Phi &= \sin(xz) - xe^{-y} + C, \end{aligned}$$

5. *Find the pressure (5p)*

a) We must have $\nabla \times \nabla p = 0$, which gives $A = 2$. It is also possible to start solving (b), and find out which A is possible.

b) $p = R(\frac{1}{2}x^2 + 2zx + y^2 + yz - z^2 + D)$, where D is an undetermined constant.

6. *Important terms*

a)

$$\begin{aligned}
[(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}]_x &= v_x\frac{\partial}{\partial x}v_x + v_y\frac{\partial}{\partial y}v_x + v_z\frac{\partial}{\partial z}v_x \\
[\nabla p]_x &= \frac{\partial p}{\partial x} \\
\left[\nabla^2\boldsymbol{v}\right]_x &= \frac{\partial^2}{\partial x^2}v_x + \frac{\partial^2}{\partial y^2}v_x + \frac{\partial^2}{\partial z^2}v_x \\
[\nabla(\nabla\cdot\boldsymbol{v})]_x &= \frac{\partial}{\partial x}(\frac{\partial}{\partial x}v_x + \frac{\partial}{\partial y}v_y + \frac{\partial}{\partial z}v_z)
\end{aligned}$$

b)

$$\begin{aligned}
\tfrac{\partial}{\partial t}v_x &= \nu\tfrac{\partial^2}{\partial z^2}v_x \\
0 &= 0 \\
0 &= -g_0 - \tfrac{1}{\rho}\tfrac{\partial}{\partial z}p
\end{aligned}$$

Chapter 2

7. *Book 2.1, 2.4, 2.5, 2.8*
2.8 before a: Constant bulk modulus K and depth z gives $p = p_0 + K\ln\frac{\rho}{\rho_0}$ and $\frac{1}{\rho} = \frac{1}{\rho_0} - \frac{g_0}{K}z$.
2.8a) Partly "show that". Critical depth: $z = \frac{K}{\rho_0 g_0}$.

8. *Book 2.2*
a) Side i with water depth d_i: $p_i(z) = p_0 + \rho_0 g_0(d_i - z)$ for $z \le d_i$ and $p_i(z) = p_0$ for $z > d_i$.
b) ca $2.7\cdot 10^6$ N.
c) ca 10^7 Nm.
d) 3.8 m (exactly).

9. ***The Standard Atmosphere (6p)***

a) $b = a/T_0$ and $c = g_0/(R_{\rm air}a)$.

b) Solving for ρ gives $\rho(z) = \rho_0(1 - bz)^{c-1}$. The numerical help in the problem gives $c \approx 5$ and $b \approx 1/(44\,\text{km})$, with other values given, it gives $\rho(11\,\text{km})/\rho(0) \sim (1-\frac{1}{4})^4 = (3/4)^4 \sim 0.3$.

c) $\gamma = c/(c-1)$

10. ***Relations between barotropic equations***

a) Incompressibility corresponds to $\gamma \to \infty$, isothermal model to $\gamma = 1$.

b) Show that. Hint: use $\gamma \to 1^+$ rather than $\gamma = 1$ for isothermal model

Chapter 3.1

11. ***Book 3.1, 3.2, 3.3***
3.2: ca 2 cm

Chapter 12

12. ***Eq. of continuity: Sources and sinks***

a) $\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) = 0$.

b) Inverse time.

c) Show that.

d) Show that.

e) Show that.

f) Inversely proportional to the square of the distance.

13. ***Flow visualization (5p)***

a) With standard x and y axes, we get horizontal lines pointing right at $t = 0$, 45° pointing up+right at $t = \frac{1}{2}$ and vertical lines pointing up at $t = 1$.

b) The trajectory starts horizontally to the right at $(0, 0)$, bends upwards roughly as a circle, and ends vertically upwards in (1/2, 1/2).

c) The streak line starts horizontally to the left at (1/2, 1/2), bends downwards roughly as a circle, and ends vertically downwards in $(0, 0)$.

14. *Book 12.2, 12.3, 12.4*

12.2:
a) Parallel lines, in the time-varying direction $(cos\omega t, \sin\omega t)$.
b) Counter-clockwise circle with radius a/ω.
c) Clockwise circle with radius a/ω.

12.4: The velocity in the 1 inch pipe is 27/32 of the velocity in the thick (3/4 inch) branch, and 3/4 of the velocity in the thin (1/2 inch) branch.

15. *Book 12.6, 12.7*

Chapter 18.0-18.2

16. *Geostrophic wind (calculator needed)*

a) TBA

b) Show that. (Note, with the precision given, both values should be 0.002 Pa/m.)

c) With $|\partial_x p| = |\partial_y p|$, it is a south-western wind (coming from south-west, velocity vector pointing north-east). Components $v_x = v_y \sim 14\,\mathrm{m/s}$ (estimated without calculator).

17. *Isobar surfaces (6p)*

a) $C = -\frac{1}{g_0} 2\Omega_z v_0$.

b) The gradient is perpendicular to contour surfaces, so $\nabla p \propto \widehat{\mathbf{e}}_z - C\widehat{\mathbf{e}}_y$ is normal to isobar surfaces, with C defined above, independently of any variation in ρ. As a side note, this problem is restricted to the situation with constant $\mathbf{v}$, which in principle restricts us to barotropic relations $p = p(\rho)$.

c) With $2v_0/g_0 \approx 8\,\mathrm{s}$, flying distance $\Delta y = -350 \cdot 10^3\,\mathrm{m}$, and altitude slightly above 45° north, so that cosine of this angle is a bit more than $\cos\pi/4 = 1/\sqrt{2}$ (shall we say 0.8?), we get $\Omega_z = 0.8 \cdot 2\pi/(24 \cdot 3600)\,\mathrm{s}^{-1} \approx 0.8/(4 \cdot 3600)\,\mathrm{s}^{-1} = (1/18000)\,\mathrm{s}^{-1}$ and $\Delta z = (8/18000) \cdot 350000\,\mathrm{m} \approx 160\,\mathrm{m}$. The final answer can be quite far away, a factor 5 or so, and still earn full score, if the used rough estimates are motivated.

18. ***Missing data (6p)***

a) With ρ the density, $\mathbf{\Omega}_z$ the local (vertical) angular velocity of the earth, $v_0 = 6\,\mathrm{m/s}$ and $L = 20\,\mathrm{km}$, the missing data is $p_B = p_A - L\rho\Omega_z v_0$.

b) In the southern hemisphere, Ω_z is negative, so $p_B > p_A$.

c) A typical length scale is $S = 500\,\mathrm{km}$ and our only available estimate of a typical velocity is v_0. The Rossby number at an angle θ from a pole of the earth is $v_0/(2S\Omega|\cos\theta|) \sim 6/(2 \cdot 5 \cdot 10^5 \cdot 0.5 \cdot 10^{-4}|\cos\theta|) = 6/(50|\cos\theta|)$. "Near" the equator, meaning $|\cos\theta| < 1/8$ or so, the Rossby number is too large for geostrophic balance to be a valid assumption. (A more trivial remark is that point A must be at least 20 km away from the south pole for point B to exist...)

19. ***Book 18.2 with hints***
The interface is 1-2 m higher on the western bank, depending on how you interprete "4%" and "25%". Compared to the example on page 314, the interface difference is about 5 times as big as the upper surface shift, and in the opposite direction.

20. ***Extra-curricular: Check section 18.5***

Chapter 13

21. ***Book 13.2, 13.3, 13.4***
13.2: ca 1 min 4 s.

22. ***Ideal flows (6p)***

a) Only case (iii) is steady.

b) Case (i) and (iii) are irrotational.

c) Case (iii) could be the velocity field described. The pressure is $p = C - (x^2 + y^2)$ where C is a constant.

23. ***Pitot by-pass (6p)***
$U \approx 2\,\mathrm{m/s}$.

24. ***Velocity Measurement (6p)***

a) The surface will be higher in the pipe attached to the wide region.

b) The height difference is $\frac{1}{20}\left(\frac{1}{0.8^2} - 1\right)\,\mathrm{m}{\sim}\ 3\,\mathrm{cm}$.

Chapter 14

25. ***Book 14.1,14.3,14.5***

Chapter 25

26. ***Book 25.3***
The group velocity scales as the square root of the depth (near beach we have shallow-water waves), so the more distant part of a wave travels faster, causing the wave to bend towards the shore, arriving more or less parallell.

27. ***How to measure a water wave***

a) Show that

b) Show that

c) show that

d) About 0.97 for the tsunami and very roughly 10^{-13} for the smaller wave. The conclusion is that the correction can be neglected for shallow water waves. Note that the tsunami is a shallow-water wave, despite being out in the ocean. That is because the ocean is shallow compared to the wavelength.

28. *Deep-water Waves (7p)*

a) Show that

b) In part show that. At $z = h$ we want $\frac{\partial v_z}{\partial t} = \frac{\partial^2 h}{\partial t^2}$, corresponding to $-a\omega^2 \exp(kh)\cos(kx - \omega t) = -\omega^2 a \cos(kx - \omega t)$. This is satisfied if $\exp(kh) \approx 1$, which introduces the condition $kh \ll 1$. (For deep-water waves, the other condition $a \ll d$ is trivially satisfied.)

c) Show that.

Chapter 15

29. *Book 15.1, 15.4*

30. *Frictionless circulation*

a) $V_{12} = \omega$, $V_{21} = -\omega$. All other $V_{ij} = 0$.

b) Show that.

31. *A pipe*

a) Show that.

b) $0 = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right)$, $0 = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}$, $0 = -g_0 - \frac{1}{\rho_0}\frac{\partial p}{\partial z}$.

c) Show that.

d) Show that.

e) Dimension of G is that of $\nabla^2\mathbf{v}$, so 1 over length and time. Since G depends only on different length scales and $\langle v_x\rangle$, the average velocity is the only variable that can give time dimension in the denominator. To do so, G must be linear in $\langle v_x\rangle$.

32. ***Shear wave***

33. ***Sound attenuation***

34. ***Book 15.6, revised and with hints***

Chapter 18.3

35. ***Ekman layer: Check equation 18.27***

a) Show that.

b) Show that. Do not forget the boundary $z \to \infty$

c) Show that.

36. ***Exam 2015-06-05, problem 5 "The Gulf Stream"***

a) The Coriolis force in geostrophic balance will act to the right for streams in the northern hemisphere, which follows from the term $-2\Omega_z \widehat{\boldsymbol{e}}_z \times \boldsymbol{v}$. The pressure at the water surface is expected to be fairly constant, and a height difference will then create a pressure gradient which can componesate the Coriolis force. To do so, the surface must be higher in the right edge of the stream. For counter-clockwise motion, that is the outer edge. (This correponds to winds going counter-clockwise around a low pressure. The lower surface on the inner side creates a "low pressure" for water beneath the surface. The current goes counter-clockwise around that low pressure.)

b) The simplest argument is that viscosity will change the velocity to become smaller, and then the force from the pressure gradient will dominate. So there will be a velocity component in the direction of negative pressure gradient, inwards. This will reduce the height difference between the outer and inner edge.

c) The Coriolis effect still drives the flow to the right, so now the inner edge must be the higher (a clock-wise flow around a "high pressure" in the middle), with the same difference magnitude as before. The Ekman layer at the bottom will be influence by the new pressure gradient and go outwards. It will still reduce the height difference.

d) We want to compare the "left-ward" (centripetal) acceleration the Coriolis acceleration effect, so we look at the Rossby number $U^2/R2\Omega U = U/2\Omega R$. A typical U was given as $1\,\mathrm{m/s}$ and the typical radius can hardly be smaller than the width of the stream, so a reasonable upper bound on the Rossby number $\sim 1/(10^{-4}10^{5}) = 0.1$. A student cleverly pointed out that these 10% cooperate with the Coriolis effect for counter-clockwise flow, but counter-acts it for clockwise flow, so the magnitude of the height difference in the two scenarios is not quite the same. A beautiful remark, which definitely is not needed for full score.

37. *Exam 2016-06-03, problem 3 "Barotropic, Geostrophic Catastrophe"*

a) The chain rule $\frac{1}{\rho}\nabla p = \omega'(p)\nabla p = \nabla\omega$ and the gravitational potential rewrites the geostrophic balance to $0 = -\nabla\Phi - \nabla\omega - 2\boldsymbol{\Omega}_z \times \boldsymbol{v}$. Taking the curl of this, and using $\nabla \times \nabla = 0$ we get the result.

b) First x component: $0 = [\nabla \times (\hat{\mathbf{e}}_z \times \boldsymbol{v})]_x = \frac{\partial}{\partial y}[\hat{\mathbf{e}}_z \times \boldsymbol{v}]_z - \frac{\partial}{\partial z}[\hat{\mathbf{e}}_z \times \boldsymbol{v}]_y$. Since a cross product involving $\hat{\mathbf{e}}_z$ is perpendicular to the z axis, the first term vanishes, and we get $0 = -\frac{\partial}{\partial z}[\hat{\mathbf{e}}_z \times \boldsymbol{v}]_y = -\frac{\partial}{\partial z}(v_x - 0)$ so $\frac{\partial}{\partial z}v_x = 0$. Similarly, the y component of the full epxression gives $\frac{\partial}{\partial z}v_y = 0$.

c) Most important is to consider friction (or viscosity), which creates a boundary layer interpolating between ground and a region of geostrophic balance. The key "qualitative" way it affects the wind is that it creates a component of the wind going (on average, in case of turbulence) in the direction of the pressure gradient force.

The topics above were considered enough for a full score. If some of it was lacking, points could be rewarded for more subtle information, *e.g.*: if the simple assumptions of Ekman holds, the wind as a function of height will "spiral"; to explain data, we must include the contribution of turbulence on the effective viscosity.