Comments on the litterature for chapter 13

Extra-curricular parts
• Section 13.2.
• Most of 13.4-5. The only message I picked up is that "vorticity" $\nabla \times \boldsymbol{v}$ is important, and determines how the Bernoulli field behaves. Remembering that, you may treat 13.4-5 as extra-curricular.
• Stream functions (latter half of 13.6, starting at p.221). You will notice quite a few old exam questions making use of the stream function, but I did not have time to lecture it this year, and will avoid such problems. If you want to try them, you can easily grasp the stream function through self studies.
• 13.8. Going from "infinitely extended cylinders" to more realistic spheres makes the math a bit longer, but does not change any significant conclusions. I focused on 13.7.

Steady, Incompressible
The Euler equations (13.1) are important. Make sure you feel comfortable with picking out, say, the z-component of $\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \boldsymbol{g} - \frac{1}{\rho_0}\nabla p$. Furthermore, in order to solve problems, it is almost always necessary to include the equation of continuity, in the form $\nabla \cdot \boldsymbol{v} = 0$ or macroscopically as Leonardo's law. Remembering the boundary conditions is also important for problem solving.

The chapter introduces the powerful Bernoulli field

$$\boldsymbol{H} = \frac{1}{2}\boldsymbol{v}^2 + g_0 z + p/\rho_0,$$

which according to Bernouilli's theorem is **constant along stream lines**. (In this note I assume $\boldsymbol{g} = (0,\ 0,\ -g_0)$, so that $\Phi = g_0 z$.) Instead of integrating Euler's equations again for every new problem, $\boldsymbol{H}$ gives us the resulting relation between pressure, position and velocity. We can interprete it as one energy-per-mass part $(\frac{1}{2}\boldsymbol{v}^2 + g_0 z)$ and one work-done-per-mass part (p/ρ_0). The latter term represents the fact that a material particle is not isolated and can perform work (or be worked upon) by the surroundings.

Applications
Bernoulli's theorem has many very nice everyday applications. We went through the three in the book: Toricelli's law, the Venturi effect and the Pitot tube.

Steady, Incompressible, Irrotational
When the flow is irrotational (here we need to remember the importance of vorticity $\nabla \times \boldsymbol{v}$) we can make some simplifications. To begin with $\boldsymbol{H}$ becomes a global constant for steady flow.

Furthermore, we may represent the vector field $\boldsymbol{v}$ with a velocity potential Ψ. Then we get the slightly easier tasc to find a scalar field, instead of a vector field, and can later get the velocity from $\boldsymbol{v} = \nabla\Psi$.

Flow around cylinder
We went through the exampel of flow around a cylinder of radius a, with "asymptotically

uniform wind", meaning that the wind is $\boldsymbol{v} = (U,\ 0,\ 0)$ "far away" from the cylinder.

We worked in cylindrical coordinates (r, ϕ) defined by $x = r\cos\phi$, $y = r\sin\phi$. Incompressible flow implies $\nabla^2\Psi = 0$. I did not dare to jump as quickly to the solution as is done in the book. Instead I started with the general solution

$$\Psi(r, \phi) = \sum_n \left(A_n r^n + B_n \frac{1}{r^n}\right)\cos(n2\pi\phi) + \left(C_n r^n + D_n \frac{1}{r^n}\right)\sin(n2\pi\phi)$$

I applied the boundary condition at $r = a$ first, $\frac{\partial\Psi}{\partial r}\Big|_{r=a} = 0$ and we found that it restricts the solution to

$$\Psi(r, \phi) = \sum_n A_n \left(r^n + \frac{a^{2n}}{r^n}\right)\cos(n2\pi\phi) + C_n \left(r^n + \frac{a^{2n}}{r^n}\right)\sin(n2\pi\phi).$$

Now, we see that we cannot include any term scaling like r^{-n} unless we also include one scaling like r^n. The asymptotic condition $\lim_{r\to\infty}\Psi = Ux = Ur\cos\phi$ then gives eq. (13.52).

Then, making use of our beloved $\boldsymbol{H}$ we could find the pressure at the surface of the cylinder, and found out that there was no lift, and no drag, on it. It follows from the high symmetry of the problem which makes the pressure mirror symmetric both in $x = 0$ and $z = 0$.

Half cylinder at bottom
The cylinder solutions satisfy the boundary conditions for a bottom at $z = 0$, so we can apply the result directly. With pressure p_∞ in locations with $|\boldsymbol{v}| = U$, we can find the total force on the top of a half cylinder lying on the bottom. To get the equations for the lifting force given in the book, we must assume that the bottom applies the pressure p_∞ from below. That is reasonable, but not obvious.

So, instead of discussing numerical values of of the lifting force, I gave an alternative, purely qualitative argument for its existence. The stream lines bend upwards (fluid accelerates upwards) above the bottom, but bends downwards above the object (the half cylinder). Applying Newton's 3rd law as a local phenomenon, we then expect an upward force on the object and a downard force on the bottom. The bottom has infinite mass and does not move, but the object may.

With these discussions on the white board, it was impossible not to debunk the myth about air plane wings. The shape of a wing is not primarily designed to make use of Bernoulli's theorem, but rather (in a simplistic description) to help accelerate the fluid downwards with a minimum of turbulence. The interested reader can have a look at chapter 29, but it is not at all part of the course.

Inlet-outlet assymetry
At the end we discussed a general feature in ideal flow problems. Lacking boundary conditions for $\boldsymbol{v}$ along walls, and having no friction (that introduces an "interaction" between neighbouring stream lines with different $\boldsymbol{v}$), makes most problems under-determined. Finding $\boldsymbol{v}$ only using the geometry of the problem, is then beyond reach. (Instead, the everyday applications start with a "common sense" assumptions about $\boldsymbol{v}$ and look for p or some other property.)

When we assume irrotational flow we get the extra constraints we need to determine $\boldsymbol{v}$ from the geometry. However, in a symmetric problem like the cylinder in uniform wind, the flow behind the object must then be the mirror image of the flow in front. The "common sense" assumptions would rather be to introduce a **trailing wake**, a region behind the object where $\boldsymbol{v} \approx 0$.

The need to make extra assumptions is in the book discussed in 13.9 (about the trailing wake) and in 13.3 (about inlet-outlet assymetry). Since those two examples are a bit far apart, I stressed that they represent the same need to sneak in viscosity through some assumptions about an assymetric $\boldsymbol{v}$. Typically, that means that the flow is not entirely irrotational. It may however be irrotational in a large region (e.g. in front of the flying object as in 13.9, or before the narrower region in a canal as in figure 13.2)