Comments on the litterature for chapter 14+25

Small time-varying perturbations
In both chapters, we focused on small, time-varying perturbations to steady flow. We saw that it leads to oscillations. In both cases, "small" was specified to mean that we could neglect the advective term, or in mathematics: $|(\boldsymbol{v} \cdot \nabla)\boldsymbol{v}| \ll |\partial \boldsymbol{v}/\partial t|$. We worked with perturbations to the simplest possible steady flow, $\boldsymbol{v} = 0$.

Extra-curricular parts chapter 14
- Page 232.
- Page 235 and onward.

Sound waves in hydrostatic equilibrium
Pages 230-231 are important. How to linearize the equations resembles techniques in many different physics disciplines. We learnt that the speed of sound c_0, is determined by a pressure-density relation at equilibrium, and found agreement with sound speed in air. Also important is the conclusion that "small" $\boldsymbol{v}$ must satisfy $|\boldsymbol{v}| \ll c_0$.

Extra-curricular parts chapter 25
- page 430 and onwards. Pages 430-431 make nice reading though, if you are willing to accept the properties of "surface tension" that we have not covered.

The mechanisms driving surface gravity waves
The introduction of the chapter 25 is important, introducing amplitude a, wavelength λ, angular frequency ω, depth d, surface $h(x, t)$, pressure, density and gravity. It also clarifies the interaction between surface, pressure, and (near) incompressibility, in creating oscillations around equilibrium.

Quick dimensional arguments
Pages 420-421 give some beautiful arguments, combinining dimensional relations and order-of-magnitude estimates to find typical behaviours of a surface wave. However, I would not be surprised if a student found those pages more enjoyable after having done the more mathematical derivation. You would then follow the typical path of doing math (p.424-428) and then take a step back to see if the solution makes sense (p.420-421).

Celerity and group velocity
The two concepts are important. Celerity is often called "phase velocity". It describes the velocity of a wave crest ("top"). The group velocity instead describes the motion of the front and back of the whole wave package. I discussed the waves behind a boat as an example.

General solution
As an alternative to the book (p.424-428), I avoided the velocity potential and was a bit more careful in how we find the correct linear combination of solutions to the Laplace equation.

First, just to slim notation, we introduced a "generalized potential" $\Phi^* = g_0 z + p/\rho_0$. For incompressible flow (which we work with), Φ^* solves the Laplace equation:
$\nabla^2 \Phi^* = \nabla \cdot (\nabla \Phi^*) = -\nabla \cdot \frac{\partial \boldsymbol{v}}{\partial t} = -\frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{v}) = 0$. Many boundary conditions restrict v_z, and

it is convenient to work with the function

$$\zeta = \frac{\partial \Phi^*}{\partial z} = -\frac{\partial v_z}{\partial t}. \tag{1}$$

The function ζ also solves the Laplace equation ($\nabla^2 \frac{\partial \Phi^*}{\partial z} = \frac{\partial}{\partial z}\nabla^2 \Phi^* = 0$) and satisfies the boundary conditions

$$\zeta(z = h) = -\frac{\partial^2 h}{\partial t^2} = \omega^2 a \cos(kx - \omega t) \tag{2}$$

$$\zeta(z = -d) = 0 \tag{3}$$

We looked for simple harmonic waves, $h = a\cos(kx - \omega t)$, knowing that in a linear problem, a more general periodic wave can be a superposition of such h-functions, for different $k_n = n2\pi/\lambda$, $n = 1, 2 \ldots$. The general solution to the Laplace equation can be written as a linear combination of functions $\exp(\pm\tilde{k}z)\cos(\tilde{k}x)$ and $\exp(\pm\tilde{k}z)\sin(\tilde{k}x)$, for different $\tilde{k}$. The boundary condition at $z = -d$ shows that the terms for $\exp(\tilde{k}z)$ and $\exp(-\tilde{k}z)$ must be combined to $\sinh[\tilde{k}(z + d)]$, which is 0 at $z = -d$. The boundary condition at $z = h$ then shows that only $\tilde{k} = k$ can be allowed, and that the trigonometric functions must be combined to $\cos(kx-\omega t)$. We now want to find a constant A that solves $A\sinh[k(h+d)]\cos(kx-\omega t) = \omega^2 a\cos(kx - \omega t)$. However, $A = \frac{\omega^2 a}{\sinh[k(h+d)]}$ depends on h, which depends on x. So A is not constant, as it must be to solve the Laplace equation. What went wrong? Even though our differential equation is linear, our boundary conditions are not: they depend on h which in turn depends on the solution.

Since we have already assumed "small amplitude" waves, we have a way to get around this. We have $|kh| \leq |ka|$ and let "small amplitude" mean $|ka| \ll 1$ so that $\exp(\pm kh) \approx 1$. Then we get the solution $A = \frac{\omega^2 a}{\sinh(kd)}$. The integral $\Phi^* = \int \zeta \mathrm{d}z$ then gives

$$\Phi^* = \omega^2 a \frac{\cosh[k(z + d)]}{k\sinh(kd)} \cos(kx - \omega t) + B(x, t). \tag{4}$$

Assuming constant air pressure p_0 at $z = h$ gives $\Phi^*(z = h) = g_0 h + p_0/\rho_0$. Again using $\exp(\pm kh) \approx 1$, we find $\Phi^*(z = h) \approx \Phi^*(z = 0)$. This determines $\Phi^*(z)$ and gives us the important *dispersion relation*

$$\omega^2 = g_0 k \tanh(kd). \tag{5}$$

From $\nabla\Phi^* = -\frac{\partial \boldsymbol{v}}{\partial t}$ we can then integrate over t to find $\boldsymbol{v}$ as in the book, up to a time-independent term $\boldsymbol{v}_0(x, z)$. We were looking for small-amplitude perturbations around a steady flow, and we have now found that we do not have many constraints on the steady flow $\boldsymbol{v}_0$. Since $\nabla \times \boldsymbol{v}_0 \neq 0$ does not destroy any arguments in our derivation, the solutions we have found for the time-dependent part do not require the assumption of irrotational flow. After that, we did not discuss the steady flow $\boldsymbol{v}_0$ further.

Two limits

In the *deep water limit*, $kd \gg 1 \Rightarrow d \gg \lambda$, we get $\cosh(kd + \alpha) \approx \exp(kd + \alpha)$ and the same for $\sinh(kd + \alpha)$. (In our equations, we have $\alpha = 0$ or $\alpha = kz$.) The solutions then turn out to be independent of d. That is reasonable: when the bottom is really far away, it does not matter for the surface wave how far "really far" is. We also find the dispersion relation $\omega^2 = g_0 k$ so that the celerity becomes $c = \sqrt{g_0/k}$ and the group velocity becomes $c_g = c/2$. The fact that $c_g < c$ is something you can see behind a speeding boat: the wave crests extend in a different angle than the wave package itself. The book makes an additional, beautiful observation about waves of different wavelengths reaching the shore at different times during the progression of a storm.

In the *shallow water limit*, $kd \ll 1 \Rightarrow d \ll \lambda$, it is safest to keep the full solutions including hyperbolic functions. The dispersion relation can still be simplified, using $\tanh(kd) \approx kd$, to get $c = \sqrt{g_0 d}$ and $c_g = c$.