Comments on the lectures for chapter 18

18.0-18.2

Local angular velocity

The main achievment in the beginning of the chapter is perhaps not to reach eq. (18.7), but to then approximate it!

The centrifugal force can be absorbed into the acceleration field $\mathbf{g}$, with negligible numerical consequences. As I discussed on the lecture, including the centrifugal term into $\mathbf{g}$ will make $\mathbf{g}$ point vertically to the local earth surface.

The angular velocity vector $\boldsymbol{\Omega}$ has one vertical component $\boldsymbol{\Omega}_z$ and one horizontal $\boldsymbol{\Omega}_y$, pointing north. The effects of the horizontal $\boldsymbol{\Omega}$ component can also be neglected in most applications (and always in this course). The cross product $\boldsymbol{\Omega}_y \times v_x\widehat{\mathbf{e}}_x$ points in the vertical direction, and is usually negligible compared to $\mathbf{g}$. The cross product $\boldsymbol{\Omega}_y \times v_z\widehat{\mathbf{e}}_z$ points in the west-east direction and is negligible compared to other terms, because "v_z tends to be small". That is a fair statement in the atmosphere and ocean applications discussed in this chapter.

The only ficticious acceleration field we need to consider is therefore $-2\boldsymbol{\Omega}_z \times \mathbf{v}$.

Important: Until eq. (18.7), the book uses $\boldsymbol{\Omega}$ for the total angular velocity. In the next section and onwards, it instead uses notation $\boldsymbol{\Omega}_0$ for the total, and lets $\boldsymbol{\Omega}$ denote the local angular velocity. In lecture notes and extra problems, I will use notation $\boldsymbol{\Omega}_z$ for the local angular velocity vector.

Rossby number

Very important and beautiful is the introduction of the Rossby number, that allows us to neglect the non-linear advective term. Horray! This means that we will not apply eq (18.10) anywhere, but immediately approximate it into (18.12).

Equation (18.12) is the fundamental equation for **geostrophic balance**, written in a way independent of coordinate system. (When we use notation $\boldsymbol{\Omega}_z$, we have specified a z-direction, though.) As often, this general equation is a bit far from finding answers to specific problems. The text, from (18.12) to the example on Great Danish Belt, illustrates how to go from the general equation to more specific (but also not generally valid!) equations.

I did not have time to show the Taylor-Proudman theorem. The derivation involves some gymnastics with ∇ and cross products. I hope to return to this when we talk about the Ekman layer.

18.3 Ekman Layer

Extra-curricular parts

Anything marked with *. Also, the Taylor columns (bottom of p. 314). The section leading to eq. 18.27 need not be reproduced, but the student is expected to be able to confirm that the equation is a solution to its problem, and analyze its properties.

Since this is an introductory course, we tend to focus on very simple systems, where boundary conditions are easily treated. Trying to do that with geostrophic flow, we reach strange results (there cannot be any wind, for example). Thus, the Ekman layer is our representative of an important procedure in fluid dynamics, where the simple solutions are applied in their respective domain, and a boundary layer that interpolates between the solutions is studied in more detail afterwards. If the boundary layer is found to have reasonable thickness, we confirm a posteriori that the procedure was valid.

The lectures followed a slightly different path than the book:

We look over distances where the flow can be assumed fairly independent of horizontal coordinates. In geostrophic balance we then have constant pressure gradient and velocities

$$\begin{aligned} \boldsymbol{g} - \frac{1}{\rho}\nabla p &= (a_x,\, a_y,\, 0), && (1)\\ \boldsymbol{v} &= (v_x,\, v_y,\, 0). && (2) \end{aligned}$$

We now add viscosity to geostrophic balance:

$$0 = \boldsymbol{g} - \frac{1}{\rho}\nabla p - 2\boldsymbol{\Omega}_z \times \boldsymbol{v} + \nu\nabla^2\boldsymbol{v} + \hat{\nu}\nabla(\nabla\cdot\boldsymbol{v}), \tag{3}$$

and assume that the only change we need to consider is the one we are interested in: the height dependence of the flow. Thus we keep eq. (1) but modify the velocity field to

$$\boldsymbol{v}(z) = (v_x(z),\, v_y(z),\, 0). \tag{4}$$

The horizontal components of our equation of motion become

$$\begin{aligned} 0 &= a_x + 2\Omega_z v_y + \nu v_x'', && (5)\\ 0 &= a_y - 2\omega_z v_x + \nu v_y''. && (6) \end{aligned}$$

Introducing

$$\begin{aligned} u_x(z) &= v_x(z) - \frac{a_y}{2\Omega_z}, && (7)\\ u_y(z) &= v_y(z) + \frac{a_x}{2\Omega_z}, && (8)\\ \delta &= \sqrt{\frac{\nu}{\Omega_z}}, && (9) \end{aligned}$$

we find

$$\begin{aligned} u_x'' &= -\frac{2}{\delta^2}u_y && (10)\\ u_y'' &= \frac{2}{\delta^2}u_x && (11) \end{aligned}$$

which combine to the fourth-order equation

$$u_y^{(4)}(z) = -\frac{4}{\delta^4}u_y(z). \tag{12}$$

Since $(-1)^{1/4} = \frac{1+i}{\sqrt{2}} \exp(in\frac{\pi}{4})$ for integer n, we find four independent solutions that after some book-keeping can be combined to

$$u_y = \exp(\frac{z}{\delta})[A\cos\frac{z}{\delta} + B\sin\frac{z}{\delta}] + \exp(-\frac{z}{\delta})[C\cos\frac{z}{\delta} + D\sin\frac{z}{\delta}], \tag{13}$$

with four undetermined constants A, B, C, D. From eq. (10) we then find

$$u_x = \exp(\frac{z}{\delta})[B\cos\frac{z}{\delta} - A\sin\frac{z}{\delta}] + \exp(-\frac{z}{\delta})[-D\cos\frac{z}{\delta} + C\sin\frac{z}{\delta}]. \tag{14}$$

Example: Frithiof Nansen in the arctic

Frithof Nansen is standing on floating ice in the arctic and the wind blows in the x direction. How does he move?

For simplicity, assume hydrostatic equilibrium at large depths: $\boldsymbol{v}(z \to -\infty) = 0$, which implies $a_x = a_y = 0$. Thus, we must have $\boldsymbol{u}(-\infty) = 0$ and $C = D = 0$.

The two remaining boundary conditions are at the surface, related to the shear forces from the wind. Since the wind is in the x direction, we have no shear force in the y direction, which implies

$$v_y'(0) \propto \sigma_{yz} = 0. \tag{15}$$

From $v_y'(0) = u_y'(0) = \ldots = \frac{1}{\delta}(A + B)$ we can replace A and B by a single undetermined constant $-\frac{1}{\sqrt{2}}U$ (carefully chosen for future cosmetic reasons) and write

$$v_y(z) = \frac{1}{\sqrt{2}} U \exp(\frac{z}{\delta})[\sin\frac{z}{\delta} - \cos\frac{z}{\delta}] = U\exp(\frac{z}{\delta})\sin(\frac{z}{\delta} - \frac{\pi}{4}). \tag{16}$$

From eq. (10) we get

$$v_x(z) = U\exp(\frac{z}{\delta})\cos(\frac{z}{\delta} - \frac{\pi}{4}). \tag{17}$$

The final constant U can be related to the shear force in the x direction, but we already have what we need to find the direction of Nansen's motion. At depth z, the current has an angle $\theta(z)$ relative to the x direction given by

$$\tan\theta = \frac{v_y}{v_x} = \tan(\frac{z}{\delta} - \frac{\pi}{4}). \tag{18}$$

Nansen on the ice at $z = 0$ thus moves 45 degrees to the right of the wind, in qualitative agreement with the explorer's own observations. (On the southern hemisphere, Ω_z is negative, which requires a re-definition of δ, and eventually a 45 degrees leftward direction.)

Example: Wind above ground

This example is covered in the book.

Far above ground, we assume a geostrophic wind in the x direction $\boldsymbol{v}(\infty) = (U, 0, 0)$. For geostrophic balance to hold in this region, we require $\boldsymbol{g} - \frac{1}{\rho}\nabla p = 2\boldsymbol{\Omega}_z \times \boldsymbol{v} = (0, 2\Omega_z U, 0)$ which means $a_x = 0$, $a_y = 2\Omega_z U$ so that $u_y = v_y$, $u_x = v_x - U$. Thus $\boldsymbol{u}(\infty) = 0$ and $A = B = 0$.

The no-slip boundary condition at $z = 0$ implies $\boldsymbol{u}(0) = (-U, 0, 0)$, so that $C = 0$ and $D = U$. Thus

$$v_x = U[1 - \exp(-\frac{z}{\delta})\cos\frac{z}{\delta}] \tag{19}$$

$$v_y = U\exp(-\frac{z}{\delta})\sin\frac{z}{\delta} \tag{20}$$

The direction of the wind as function of height is not so easy to find, but near ground (meaning $z \ll \delta$) we use $1 - \exp(-\varepsilon)\cos\varepsilon = 1 - (1 - \varepsilon + \mathcal{O}(\varepsilon^2))(1 + \mathcal{O}(\varepsilon^2)) = \varepsilon + \mathcal{O}(\varepsilon^2)$ and $\exp(-\varepsilon)\sin\varepsilon = [1 + \mathcal{O}(\varepsilon)][\varepsilon + \mathcal{O}(\varepsilon^3)] = \varepsilon + \mathcal{O}(\varepsilon^2)$ to find

$$\frac{v_y(0)}{v_x(0)} = 1. \tag{21}$$

The wind near ground is directed 45 degrees to the left (on the northern hemisphere) of the geostrophic wind higher up.

Note that the wind near ground to some extent is moving in the direction of the negative pressure gradient. Near ground, the boundary conditions reduce the wind and therefore the Coriolis effect, so that the pressure gradient “wins”. This is similar to the tee-leaf effect when you stir your tea, as we demonstrated with map-pins in a bucket of water. There, it was a competition between centrifugal and pressure forces that was “won” by pressure near the bottom of the bucket.