Comments on chapter 2

Sections marked with star $*$ are extra-curricular.

The scalar pressure field p gives rise to contact forces on a body (or material particle), perpendicular to the contact surface.

For completeness, the chapter introduces pressure force (inward push), tension force (outward pull) and shear forces (forces along the contact surface). If you feel that shear is not well explained (in book or lectures) please note that this chapter does not consider any friction or viscosity. Only when we have introduced that, can we get good examples of shear forces. It will come...

Two fundamental equations build up the chapter. First, the total pressure force $\mathrm{d}\mathbf{F}_P$ acting on a small test-volume $\mathrm{d}V$ will involve differences of pressure forces on opposite sides, and can be written as a differential relation (*local formulation*)

$$\mathrm{d}\mathbf{F}_P = -(\nabla p)\mathrm{d}V.$$

(This holds even if the test volume is not a nice cuboid.)

Second, under **hydrostatic equilibrium** (fluid at rest), this force must balance all other forces acting on the material inside volume. If the only such force is gravitation, the equilibrium can be written

$$\nabla p = \rho \mathbf{g},$$

where ρ is the density and $\mathbf{g}$ is the gravitational acceleration, recoginized in the normal gravitational force $\mathbf{F}_G = m\mathbf{g}$ for a particle with mass m.

These equations are the starting point for a set of models, used to relate the theory to plenty of everyday phenomena.

All models in this chapter assume constant gravitational field ("flat earth"), and define the z-axis as pointing upwards: $\mathbf{g} = (0, 0, -g_0)$. Hydrostatic equilibrium now reads

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0, \quad \frac{\partial p}{\partial z} = -\rho g_0.$$

This is one equation, with 3 unknown variables (ρ, p and z), so we lack information to solve the system. The needed information can come from fundamental thermodynamics, model assumptions, or parametrization of data.

In general, thermodynamics will provide an equation of state, $F(p, \rho, T) = 0$. This gives a second a relation, but also a new unknown field, T. So we still lack one relation. With clever enough additional assumptions, we can find such an extra relation, eliminate T from F to find a **barotropic equation of state**

$$F(p, \rho) = 0.$$

From such a relation, we can hopefully find $\rho = \rho(p)$ and/or $p = p(\rho)$. (Not with mathematical necessity, but for reasonable models $F(p, \rho)$, we can.) As apparent in the chapter, it is often possible to directly assume a barotropic equation (and later on, if needed, figure out what it meant for the temperature).

Much of the chapter, spread around both before the fundamental equations, between them, and after them, is about reasonable assumptions that help us "get rid of T".

Model 1: Constant density (*incompressible fluid*).
We do not have to introduce T at all, but immediately use $\rho = \rho_0$. (Expressed in terms of F, we assume $F(p, \rho) = \rho - \rho_0$.) A decent approximation for water, but more shaky for gases.

The chapter discusses a phenomemonlogical model for *bulk modulus*, which improves the constant density model for water. The bulk modulus model as such is not essential now, but the concept of bulk modulus will appear when we cover sound waves, so the section is recommended for self-studies.

Model 2: Ideal gas in constant temperature (*isothermal*).
We combine the ideal gas law $pV = Nk_BT$ with the assumption $T = T_0$. We eliminate T and find $p = C\rho$ as our barotropic relation.

Model 3: Polytropic atmosphere
We assume that the relation between p and ρ can be written $p = C\rho^\gamma$, where C and γ are constants that must be measured. Note that C has strange dimensions, in particular for non-integer γ, so a way to write the relation is $\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^\gamma$, where boundary conditions are expressed by p_0 and ρ_0.

Special case of Model 3: **Homentropic atmosphere**.
Assuming equilibrium, there is no net flow of gas between altitudes. Due to fluctuations, material flow still occurs, with compensating flow in opposite directions. For an ideal gas, it is not possible for the compensating flow of material to also compensate any non-zero transfer of heat. The assumption is therefore that under equilibrium, material flow between altitudes is *adiabatic* (without heat transfer), and *isentropic* (reversible, preserving entropy). This leads to the barotropic relation $p = C\rho^\gamma$, where the Poisson constant is $\gamma = 7/5$ for diatomic gases. Thus, in this atmospheric model, γ is known from theory.

Model 4: Standard atmosphere (1976)
While the other models combine a fundmaental thermodynamic relation with some extra model assumption, the standard 1976 atmosphere adds a paramterization of measurements. This does not give any additional understanding to the measurements, but it does provide more accurate results to plug into other applications. More precisely the model parametrizes $T'(z)$, which together with ground temperature T_0 determines $T(z)$. That can be used to eliminate T from a general relation $F(p, \rho, T) = 0$.

Most important in this chapter is perhaps to understand the *general strategy*. Find enough relations between p, ρ, and possibly T, to be able to eliminate all but one and find how that depends on z. Since hydrostatic equilibrium involves differentiation, some boundary condition must be introduced to determine integration constants.