Extra exercises chapter 13 & 14

1. Steady flows

Consider the two-dimensional steady flows

$$\boldsymbol{u}_1 = (2y, -x, 0), \qquad \boldsymbol{u}_2 = (x, -y, 0).$$

a) Show that $\nabla \cdot \boldsymbol{u}_i = \boldsymbol{0}$, $i = 1, 2$.

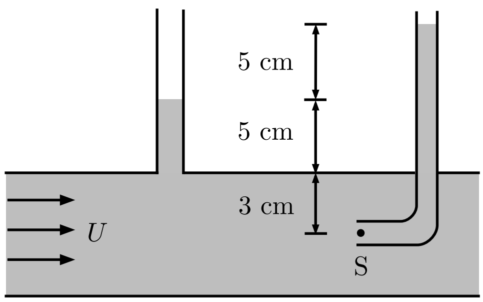
b) Compute the vorticity $\boldsymbol{\omega}_i$, $i = 1, 2$.

c) Find stream functions ψ_1, ψ_2 and draw the streamlines.

d) If possible, calculate velocity potentials Ψ_1, Ψ_2.

2. Pitot tube [From Exam 13.08.29]

The figure below illustrates a steady water flow in a pipe. The water is in contact with the air through a piezometer tube at the top of the pipe and a Pitot tube which is lowered into the pipe. The Pitot tube is directed towards the flow and the point S at the entrance is a stagnation point where the fluid velocity is 0. The water is assumed to be completely still in both tubes. At the entrance to the pipe, the flow is assumed to be uniform with speed U. Determine U by using the information in the figure and making reasonable assumptions. You can use the approximations $g \approx 10$ m/s^2 and $\rho \approx 1000$ kg/m^3, where ρ is the density of the water.



3. A vector identity

In the proof of eq. (13.24) on p. 217 we used the identity

$$\boldsymbol{v} \cdot (\nabla_i \boldsymbol{v}) - (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}_i = (\boldsymbol{v} \times (\nabla \times \boldsymbol{v}))_i.$$

Verify this identity.

4. Spherical waves

The general solution of the one-dimensional wave equation

$$\partial_t^2 u = c_0^2 \partial_x^2 u,$$

where $u = u(x,t)$, is given by

$$u(x,t) = f(x - c_0 t) + g(x + c_0 t)$$

where f and g are arbitrary functions. The first term represents a wave moving at constant speed $c_0 > 0$ to the right and the second a wave moving to the left at the same speed.

a) Consider the three-dimensional wave equation

$$\partial_t^2 u = c_0^2 \nabla^2 u,$$

$u = u(\boldsymbol{x}, t)$. Assume that $u = u(r,t)$ only depends on the radius $r = \sqrt{x^2 + y^2 + z^2}$ and t. Show that the wave equation takes the form

$$\partial_t^2 u = c_0^2 \left(\partial_r^2 u + \frac{2}{r} \partial_r u \right).$$

b) Let $v(r,t) = r u(r,t)$. Show that v satisfies the one-dimensional wave equation $\partial_t^2 v = c_0^2 \partial_r^2 v$. Use this to write down a general solution formula for u. Interpret your result.