Extra exercises FYTA14, 2015

1. Operators and fields
For the following vector fields \( \mathbf{F} \), depending on the position vector \( \mathbf{r} = (x, y, z) \), examine \( \nabla \times \mathbf{F} \) to determine if \( \mathbf{F} \) can be written \( \mathbf{F} = \nabla \phi \), where \( \phi \) is a scalar field. If so, determine the form of \( \phi \).

\[
\mathbf{F} = (y, x, 0), \\
\mathbf{F} = (-y, x, 2z), \\
\mathbf{F} = (2xz^2 + y^2 - 2, 2xy + 1, 2x^2z), \\
\mathbf{F} = (z \cos(xz) - e^{-y}, xe^{-y}, x \cos(xz)),
\]

2. Relations between barotropic equations
The polytropic relation between pressure \( p \) and density \( \rho \) reads

\[
p = C\rho^\gamma,
\]

where \( C \) and \( \gamma \) are constants. As shown in the book, flat-earth gravity then results in a pressure equation

\[
p(z) = p_0 \left(1 - \frac{\gamma - 1}{\gamma h_0} z\right)^{\gamma/(\gamma-1)},
\]

where \( z \) is the altitude and \( h_0 = p_0/\rho_0 g_0 \). Measurements at a reference level \( z = 0 \) determines \( p_0 \) and \( \rho_0 \).

a) Show that both the incompressibility condition and the isothermal relation can be obtained with a proper value or limit of \( \gamma \) in \( p = C\rho^\gamma \). (In other words, show that they are just special cases of the polytropic relation.)

b) For the incompressibility and isothermal assumption, respectively, insert the found value or limit for \( \gamma \) in \( p(z) \) and show that it indeed gives the \( p(z) \) derived in the book for those cases.

3. Sources and sinks
Consider a thin layer of water flowing on a flat surface. This corresponds to flow in a two-dimensional world.

a) Write down the equation of continuity for two dimensions, using components explicitly.

Now, suppose water pours down from a tap, hitting the surface around a point \( \mathbf{r}_0 = (x_0, y_0) \). From a two-dimensional perspective, there is a “magical” production of water in this region,
a source. For simplicity we assume that it adds a mass \( \rho_0 \omega_0 A \Delta t \) to a surface of size \( A \) in time \( \Delta t \). (Comment: if water instead disappeared through a hole, so that \( \omega < 0 \), it would be called a **sink**.)

**b)** In the relation above, \( \rho_0 \) is a **surface density** (mass per area). What is the dimension of \( \omega_0 \)?

**c)** Assume an incompressible fluid so that the **volume density** (the usual density) is constant. Furthermore, assume the height of the water to be constant, so that the surface density also becomes constant, \( \rho(x,y) = \rho_0 \). Show that the amount of water that flows out through the boundaries of a small rectangle with sides \( \Delta x \) and \( \Delta y \) in time \( \Delta t \) is

\[
\Delta m = \rho_0 A \nabla \cdot \mathbf{v} \Delta t,
\]

where \( A = \Delta x \Delta y \) and the divergence is defined in two dimensions.

**d)** Show that this results in the equation

\[
\nabla \cdot \mathbf{v} = \omega(\mathbf{r}),
\]

where \( \omega = 0 \) outside the source and \( \omega = \omega_0 \) within the region of the source.

**e)** Assume radial flow out from the source. Show that, outside the source, the speed is inversely proportional to the distance of the source. (It can be shown with drawings and simple arguments, or with math, preferably using cylindrical coordinates.)

**f)** If we managed to create a source in a three-dimensional world, how would the speed (assuming radial flow) depend on the distance to the source? Compare to gravitational and electromagnetic fields!

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4. **The real world (calculator needed)**

Two ships are sailing with a speed of 10 km/hr. Ship A sails eastward while ship B sails toward northwest. Over 4 hours, the pressure change measured on board is 0.70 hPa for ship A and -1.0 hPa for ship B. Assume that the pressure field is stationary and that the horizontal derivatives of the pressure field are constant in the entire area.

**a)** Show that the horizontal derivatives of the pressure field are given by

\[
\frac{\partial p}{\partial x} = 0.00175 \text{Pa/m}, \quad \frac{\partial p}{\partial y} = -0.00179 \text{Pa/m}.
\]

**b)** We are at 43°N and the density of the air can be assumed to be 1.2 kg/m³. Determine the geostrophic wind vector in the area. Make a sketch of the pressure field and the geostrophic wind.

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5. **Problem 18.2 with hints**

Let the z direction be vertical and the flow in the x direction. Then pressure \( p(y,z) \) will depend on an equation like (18.15).
Introduce a function $f(y)$ that describes the height of the interface at position $y$. The pressure at the interface is then $p_i(y) = p(y, z = f(y))$. Calculate $\frac{dp_i(y)}{dy} = \frac{\partial p}{\partial y} + f'(y)\frac{\partial p}{\partial z}$. Now comes the trick: since you are at the interface, this equation should give the same result if you use $\rho$ and $U$ from the brackish or saline region, respectively. Use the trick to determine $f'(y)$ and then the height difference between the west and east bank.

6. **Check section 18.5**
Check the numerical estimates and the results in section 18.5.

7. **Check equation 18.27**
   a) Verify that (18.27) solves (18.23).
   b) Verify that (18.27) satisfies the stated boundary conditions.
   c) Verify the approximate direction of the wind near ground, $0 < z \ll \delta$.

8. **Shear wave**
Verify that $v_x$ in eq. (15.12) solves eq. (15.5) and satisfies the stated boundary conditions.

9. **Sound attenuation**
Verify eqs. (15.36), (15.38) and (15.39). Note that eq. (15.37) defines $\omega_0$, and so is not an equation to be verified.

10. **Problem 15.6, revised and with hints**
The purpose of this problem is to show that many of the equations we see in the book can be used to solve problems with rather general boundary conditions. If we couldn’t, what would the point of discussing the equations in the first place? The problem 15.6 is, however, quite mathematical and is here split in smaller parts with extra hints. Along the way, a misprint is corrected.

The function $f(s)$ obeys eq. (15.16),

$$f''(s) + \frac{1}{2}sf'(s) = 0$$

and satisfies the boundary values

$$f(0) = 1, \quad f(\infty) = 0.$$

(These two boundary values will be needed a bit here and there in the problem.)
As stated in the book, \( f \left( \frac{y}{\sqrt{\nu t}} \right) \) solves

\[
\frac{\partial f}{\partial t} = \nu \frac{\partial^2 f}{\partial y^2}. \tag{1}
\]

a) Verify this.

b) Show that \( f \left( \frac{y}{\sqrt{\nu (t - t')}} \right) \) also solves eq. (1).

Now we have gathered all we need to know about \( f \)! We do not need the expression for \( f \), (eq 15.17). But it is nice to see that the function exists.

The \( v_x(y, t) \) in the problem is of the form

\[
v_x(y, t) = \int_0^t h(y, t, t') dt'.
\]

This implies

\[
\frac{\partial v_x}{\partial t} = h(y, t, t) + \int_0^t \frac{\partial}{\partial t} h(y, t, t') dt', \tag{2}
\]

\[
\frac{\partial^2 v_x}{\partial y^2} = \int_0^t \frac{\partial^2}{\partial y^2} h(y, t, t') dt'. \tag{3}
\]

To be consistent with the definition of \( f \) in the book, the integrand \( h \) must be

\[
h(y, t, t') = f \left( \frac{y}{\sqrt{\nu (t - t')}} \right) \dot{U}(t'). \tag{4}
\]

(In problem 15.6, \( f \) has been written as \( 1 - f \).)

In \( h \) we have a factor \( \dot{U}(t') \) which means \( \frac{d}{dt'} U(t') \). Note that \( t \) and \( t' \) are two independent variables, so that \( \frac{\partial}{\partial t} \dot{U}(t') = 0 \). The expression \( h(y, t, t) \) means \( h(y, t, t') \) where \( t' \) happens to have the same value as \( t \).

c) Using the expression stated here for \( h \), show that \( v_x(y, t) \) solves eq. (15.5). (You do not need to do calculate any derivative of \( f \) explicitly. Instead, make use of the result in (b).)

d) Show that we fulfil the boundary condition \( v_x(y, 0) = 0 \). (This is not a trick question, it is as simple as it seems.)

e) Show that \( v_x(0, t) = U(t) - U(0) \). Since the problem states \( U(0) = 0 \), we fulfil the other boundary condition \( v_x(0, t) = U(t) \).

f) Solve the follow-up problem on circulation, 15.6(b). Do not get distracted by eq. (15.19)! Instead, copy the argument in the final paragraph of p. 249. It holds even if \( U \) depends on \( t \).
11. **Frictionless circulation**
In a Newtonian fluid, the stress tensor $\sigma$ is linear in the components of the “velocity gradient tensor” $V$, where $V_{ij} = \frac{\partial v_j}{\partial x_i}$. The most general 2-rank tensor for an isotropic fluid is then

$$\sigma = -p \mathbb{I} + \eta_a V + \eta_b V^\top + \hat{\eta} \mathbb{I} (\nabla \cdot v),$$

where $p$ is pressure, $\mathbb{I}$ is the unity matrix, $V^\top$ is the transpose of $V$.

Consider steady rotating flow with constant angular velocity $\omega = \omega \mathbf{\hat{e}}_3$, so that $v(x) = \omega \times x$.

**a)** Calculate the velocity gradient tensor for this flow.

An important feature of this flow is that there is no friction. This implies that all the frictional terms in the stress tensor $\sigma$ must vanish, so that $\sigma = -p \mathbb{I}$.

**b)** Show that this implies $\eta_a = \eta_b$. In other words, that $\sigma$ for a Newtonian fluid must be symmetric.

12. **Basic vector calculus operations**

**a)** Calculate the (3-dimensional) gradient of the following scalar fields $\phi$.

$$\phi = x^2 y + z$$
$$\phi = \sin x \cos y$$

**b)** Calculate the divergence of the following vector fields $u$

$$u = (x^2, xy + 3z, y^2)$$
$$u = (x \cos x \cos y, x \sin x \sin y, -z \cos x \cos y)$$

**c)** Calculate the rotation of the fields in (b).

**d)** With explicit calculation, verify that the rotation of the gradients obtained in (a) vanish.

**e)** With explicit calculation, verify that the divergence of the rotations obtained in (c) vanish.

**f)** Calculate the Laplacian on the first field $u$ in (b). Also, calculate the gradient of the divergence, and verify explicitly that it differs from the Laplacian.
13. How to measure a water wave
It’s not easy to determine the shape of a wave at sea. One possibility is to measure the pressure at the bottom of the ocean and try to recover the surface from these measurements. For this one needs a relation between the surface elevation and the pressure at the bottom. We assume that the water can be modelled as an ideal, incompressible and irrotational fluid with constant density $\rho_0$ and that the gravitational force is given by $\mathbf{g} = -g_0 \mathbf{e}_z$.

a) Assume that we can use a hydrostatic approximation of the pressure and that the bottom is flat and given by $z = -d$, while the surface is given by $z = h(x, y, t)$. Let $\Delta p = p|_{z=-d} - p_0$ be the pressure at the bottom minus the constant atmospheric pressure. Show that

$$h = \frac{\Delta p}{\rho_0 g_0} - d$$

b) One can improve the calculations for small-amplitude waves by using the linear theory for water waves. Assuming that the surface has the form $h = a \cos(kx - \omega t)$, one can then use the relations

$$\Psi = \frac{a \omega \cosh(k(z + d))}{k \sinh(kd)} \sin(kx - \omega t),$$

$$p = p_0 - \rho_0 g_0 \left( z - a \frac{\cosh(k(z + d))}{\cosh(kd)} \cos(kx - \omega t) \right),$$

where

$$\cosh(x) = \frac{e^x + e^{-x}}{2}.$$

Show that this gives

$$h = \left( \frac{\Delta p}{\rho_0 g_0} - d \right) \cosh(kd),$$

c) Let’s now compare the results of the two methods. Assume that $h$ has the exact form $h = a \cos(kx - \omega t)$, and that the linear theory gives a correct result for $\Delta p$. Show that the method to extract $h$ from $\Delta p$ in a) yields the surface

$$h = \frac{a}{\cosh(kd)} \cos(kx - \omega t).$$

d) How big is the scale factor $\frac{1}{\cosh(kd)}$ for a tsunami with wavelength approximately 100 km on water of depth 4 km? How big is it for a wave of wavelength 2 m and depth 10 m? What’s the conclusion? Hint: One can estimate $\cosh(x) \approx 1 + \frac{1}{2} x^2$ for $|x| < 1$ and $\cosh(x) \sim \frac{1}{2} 10^4 |x|$ for $|x| \gg 1$. 

14. *Pitot by-pass*

![Diagram of a Pitot by-pass](image)

The figure above illustrates a steady water flow in a pipe. A small tube is connected at the bottom of the pipe, and returns into the pipe as a Pitot tube. The Pitot tube is directed towards the flow and the point \( S \) at the entrance is a stagnation point where the fluid velocity is 0. The lower part of the by-pass is filled with mercury, while the rest is filled with water. The fluids in the by-pass are assumed to be completely still. At the entrance to the pipe, the flow is assumed to be uniform with speed \( U \). Determine \( U \) by using the information in the figure and making reasonable assumptions. As approximations, you may use the gravitational acceleration \( g \approx 10 \text{ m/s}^2 \), the water density \( \rho_0 \approx 1000 \text{ kg/m}^3 \) and the mercury density \( \rho_1 \approx 13.5\rho_0 \).

15. *Viscous pipe flow*

Water flows in a pipe pointing in the \( x \) direction. The cross-section of the pipe is constant, and we look for a steady solution to the flow where the velocity is only along the pipe and does not change as it travels through the pipe. In other words,

\[
\mathbf{v} = ( v_x(y, z), 0, 0 )
\]

(a) Show that the co-moving derivate of the velocity is zero.

(b) Assuming constant density \( \rho_0 \) for the water and a constant gravitational acceleration \(-g_0\) along the \( z \) direction, write down what remains of Navier–Stokes equations, when vanishing terms have been removed.

(c) Show that the pressure must be of the form

\[
p = q(x) - \rho_0 g_0 z.
\]

The function \( q \) depends only on \( x \) and may at this point be undetermined.

(d) Show that the pressure fall per length is constant along the \( x \) direction,

\[
\frac{\partial p}{\partial x} = -\eta G,
\]
where $\eta$ is the (constant) viscosity of water, and $G$ is an unknown constant of appropriate dimension.

e) The only variables that can determine $G$ are different length scales for the pipe (radius, height, width, etc.) and the average velocity $\langle v_x \rangle$ of the flow. Use dimensional arguments to show that $G$ must be proportional to $\langle v_x \rangle$. 