Exercises FYTA14, 2018

Appendix C

1. *Basic vector calculus operations*

a) Calculate the (3-dimensional) gradient of the following scalar fields ϕ.

$$\begin{aligned} \phi &= x^2 y + z \\ \phi &= \sin x \cos y \end{aligned}$$

b) Calculate the divergence of the following vector fields $\mathbf{u}$

$$\begin{aligned} \mathbf{u} &= (x^2,\, xy + 3z,\, y^2) \\ \mathbf{u} &= (x \cos x \cos y,\, x \sin x \sin y,\, -z \cos x \cos y) \end{aligned}$$

c) Calculate the rotation of the fields in (b).

d) With explicit calculation, verify that the rotation of the gradients obtained in (a) vanish.

e) With explicit calculation, verify that the divergence of the rotations obtained in (c) vanish.

f) Calculate the Laplacian on the first field $\mathbf{u}$ in (b). Also, calculate the gradient of the divergence, and verify explicitly that it differs from the Laplacian.

2. *Book C.1-C.3*

The book uses position vector $\boldsymbol{x} = (x_1,\, x_2,\, x_3)$ and vectors $\boldsymbol{u} = (u_1,\, u_2,\, u_3)$, while notes and lectures typically use $\boldsymbol{r} = (x,\, y,\, z)$ and $\boldsymbol{u} = (u_x,\, u_y,\, u_z)$. The book uses the intuitive symbol ∇_i for $\frac{\partial}{\partial x_i}$, so that $\nabla = (\nabla_1,\, \nabla_2,\, \nabla_3)$. For C.3, it is enough to consider the 1-component of the equation, $[\nabla \times (\boldsymbol{U} \times \boldsymbol{V})]_1 = [\ldots]_1$.

3. *Book C.4*

Problem C.4b is included just to highlight a slightly unusal notation adopted in the book. The "outer product" between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ creates a matrix $\boldsymbol{C}$ with elements $C_{ij} = a_i b_j$. In most litterature, they are denoted $\boldsymbol{C} = \boldsymbol{a} \otimes \boldsymbol{b}$, or $\boldsymbol{C} = \boldsymbol{a} \bigwedge \boldsymbol{b}$. In this book, the equation is written $\boldsymbol{C} = \boldsymbol{a}\boldsymbol{b}$. So "adjacency" means outer product, which actually is in line with multiplication of a scalar and a vector, as in $\boldsymbol{u} = \lambda\boldsymbol{c}$. With $\lambda = \boldsymbol{a}\cdot\boldsymbol{b}$, we get $\boldsymbol{u} = (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}$. With the book notation for outer product, we can actually drop the parenthesis, $(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c} = \boldsymbol{a}\cdot(\boldsymbol{b}\boldsymbol{c}) = \boldsymbol{a}\cdot\boldsymbol{b}\boldsymbol{c}$.

4. *Field Potentials*

For the following vector fields $\mathbf{F}$, depending on the position vector $\mathbf{r} = (x, y, z)$, examine $\nabla\times\mathbf{F}$ to determine if $\mathbf{F}$ can be written $\mathbf{F} = \nabla\phi$, where ϕ is a scalar field. If so, determine the form of ϕ.

$$\begin{aligned} a)\ \mathbf{F} &= (y,\ x,\ 0), \\ b)\ \mathbf{F} &= (-y,\ x,\ 2z), \\ c)\ \mathbf{F} &= (z\cos(xz) - e^{-y},\ xe^{-y},\ x\cos(xz)), \\ d)\ \mathbf{F} &= (y^2 + 2zx,\ z^2 + 2xy,\ x^2 + 2yz) \end{aligned}$$

Hint: Read the extra paper about solving “gradient equations”.

5. *Find the pressure (5p)*

A group of students solve a problem in fluid dynamics. Everyone agrees that the pressure $p(x, y, z)$ must fulfil

$$\nabla p = R(x + 2z,\ 2y + z,\ Ax + y - 2z).$$

Here, R is a constant of appropriate dimension, and A is a dimensionless constant. Based on other parts of the problem, one student has found $A = 1$, while another student thinks that $A = 2$.

a) [2p] Is any of the suggested values of A correct? If so, which one? (An unmotivated guess will not reward any points.)

b) [3p] Find the most general expression for $p(x, y, z)$, given the correct value of A.

6. *Important terms*

Practice to write out vector expressions using components of vectors. Use $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$, $\boldsymbol{v} = (v_x,\ v_y,\ v_z)$, and let p be a scalar field.

a) Using components, write out the x-component of the following vector expressions:

$$(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}$$
$$\nabla p$$
$$\nabla^2\boldsymbol{v}$$
$$\nabla(\nabla\cdot\boldsymbol{v})$$

b) The Navier–Stokes equation reads

$$\frac{\partial}{\partial t}\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = \boldsymbol{g} - \frac{1}{\rho}\nabla p + \nu\nabla^2\boldsymbol{v} + \hat{\nu}\nabla(\nabla\cdot\boldsymbol{v}).$$

Let $\boldsymbol{v} = (v_x(z,t),\, 0,\, 0)$, $p = p(z)$ and $\boldsymbol{g} = (0,\, 0,\, -g_0)$ and all other scalar variables constant. Using components of ∇, $\boldsymbol{v}$ and $\boldsymbol{g}$, simplify each component of the Navier–Stokes equation as much as possible (cancel all terms that must be zero).

Chapter 2

7. *Book 2.1, 2.4, 2.5, 2.8*
Problem 2.8 tests the ability to *use* the book equations on constant bulk modulus: those equations need not be memorized.

8. *Book 2.2*
This is a good example of a typical kind of physics problem, and how it can be solved. First, zoom in to a small segment of the system, and write down the physics there. Then, zoom out and add upp contributions from many small segments. That will give an integral. After that, it is time to do the math.

b: Consider a strip of the gate which has the full width $L = 12\,\mathrm{m}$ and a tiny height $\mathrm{d}z$. Use the pressure from (a) on both sides to determine the total force acting on the strip. It will depend on the vertical position z of the strip. Note that you will get a different expression when z is above the lower surface at $d_2 = 6\,\mathrm{m}$. Do the integral that adds the force of all strips.

c: Do the same trick as in (b), but this time, find the torque (moment of force) for a strip at height z.

d: This is the meaning of the question: suppose the total force found in (b) was *not* distributed over the whole gate, but acted at a single height z^*. What would this z^* be for the torque found in (c) to be correct?

9. *The Standard Atmosphere (6p)*
The temperature of the atmosphere at height z above ground is roughly

$$T(z) = T_0 - az,$$

where T_0 is the ground temperature and a is a parameter. With $a = 6.5\,\mathrm{K/km}$, this parametrization works well for the lowest 11 km of the atmosphere. The air is assumed to obey the ideal gas law, which can be written

$$p = \rho R_{\mathrm{air}} T.$$

The value of R_{air} (the "specific gas constant" for air) is $287\,\mathrm{m^2/(s^2\,K)}$.

a) [3p] Assume hydrostatic equilibrium and constant gravitational acceleration $g_0 \approx 10\,\text{m/s}^2$. Let p_0 and ρ_0 be the pressure and density at ground, respectively. Show that $p(z)$ is of the form

$$p(z) = p_0(1 - bz)^c$$

and find expressions for b and c.

b) [2p] Suppose the ground temperature is $T_0 = 286\,\text{K}$ ($\approx 13°\text{C}$). Make a rough numerical estimate of the density at height 11 km, relative to ground density – in other words, estimate the ratio $\rho(11\,\text{km})/\rho_0$. Hint: 286/6.5=44. You may use the estimate $g_0/R_{\text{air}}a \approx 5$.

c) [1p] Show that the atmospheric model is *polytropic*, $p = C\rho^\gamma$, where C and γ are constants. Find an expression for γ.

10. *Relations between barotropic equations*

The *polytropic relation* between pressure p and density ρ reads

$$p = C\rho^\gamma,$$

where C and γ are constants. As shown in the book, flat-earth gravity then results in a pressure equation

$$p(z) = p_0\left(1 - \frac{\gamma - 1}{\gamma h_0} z\right)^{\gamma/(\gamma-1)},$$

where z is the altitude and $h_0 = p_0/\rho_0 g_0$. Measurments at a reference level $z = 0$ determines p_0 and ρ_0.

a) Show that both the incompressibility condition and the isothermal relation can be obtained with a proper value or limit of γ in $p = C\rho^\gamma$. (In other words, show that they are just special cases of the polytropic relation.)

b) For the incompressibility and isothermal assumption, respectively, insert the found value or limit for γ in $p(z)$ and show that it indeed gives the $p(z)$ derived in the book for those cases.

Chapter 3.1

11. *Book 3.1, 3.2, 3.3*

Chapter 12

12. *Eq. of continuity: Sources and sinks*
Consider a thin layer of water flowing on a flat surface. This corresponds to flow in a two-dimensional world.

a) Write down the equation of continuity for two dimensions, using components explicitly.

Now, suppose water pours down from a tap, hitting the surface around a point $\mathbf{r}_0 = (x_0, y_0)$. From a two-dimensional perspective, there is a "magical" production of water in this region, a *source*. For simplicity we assume that it adds a mass $\rho_0 \omega_0 A \Delta t$ to a surface of size A in time Δt. (Comment: if water instead dissappeared through a hole, so that $\omega < 0$, it would be called a *sink*.)

b) In the relation above, ρ_0 is a *surface density* (mass per area). What is the dimension of ω_0?

c) Assume an incompressible fluid so that the *volume density* (the usual density) is constant. Furthermore, assume the height of the water to be constant, so that the surface density also becomes constant, $\rho(x, y) = \rho_0$. Show that the amount of water that flows out through the boundaries of a small rectangle with sides Δx and Δy in time Δt is

$$\Delta m = \rho_0 A \Delta t \, (\nabla \cdot \mathbf{v}),$$

where $A = \Delta x \Delta y$ and the divergence is defined in two dimensions.

d) Show that this results in the equation

$$\nabla \cdot \mathbf{v} = \omega(\mathbf{r}),$$

where $\omega = 0$ outside the source and $\omega = \omega_0$ within the region of the source.

e) Assume radial flow out from the source. Show that, outside the source, the speed is inversely proportional to the distance of the source. (It can be shown with drawings and simple arguments, or with math, preferably using cylindrical coordinates.)

f) If we managed to create a source in a three-dimensional world, how would the speed (assuming radial flow) depend on the distance to the source? Compare to gravitational and electromagnetic fields!

13. *Flow visualization (5p)*
Between time $t = 0$ and $t = 1$ (in non-dimensional variables), a 2-dimensional flow is described by

$$\boldsymbol{v} = (1 - t, \, t).$$

a) [1p] Scetch stream lines at $t = 0$, $t = \frac{1}{2}$ and $t = 1$.

b) [2p] Scetch (roughly) a particle trajectory from $t = 0$ to $t = 1$, starting at $(x, \, y) = (0, \, 0)$.

What is the endpoint at $t = 1$?

c) [2p]Scetch (roughly) a streak line from $t = 0$ to $t = 1$, going through $(x,\ y) = (0,\ 0)$ at $t = 1$.

14. *Book 12.2, 12.3, 12.4*

15. *Book 12.6, 12.7*
Two very similar problems. The constraints $|\mathrm{d}h/\mathrm{d}x| \ll 1$ and $|\mathrm{d}a/\mathrm{d}x| \ll 1$ allows the approximation $v_x^2 \approx \boldsymbol{v}^2$. In each sub-problem (b), assume that $v_x(x, y, z)$ equals the average, $\langle v_x(x) \rangle$.

Chapter 18.0-18.2

16. *Geostrophic wind (calculator needed)*
Two ships are sailing with a speed of 10 km/hr. Ship A sails eastward while ship B sails toward northwest. Over 4 hours, the pressure change measured on board is 0.70 hPa for ship A and −1.0 hPa for ship B. Assume that the pressure field is stationary and that the horizontal derivatives of the pressure field are constant in the entire area.

a) Introduce trajectories for each boat, and velocities. Find expressions for $\frac{\mathrm{d}p_{boat}}{\mathrm{d}t}$ in terms of boat velocity $\boldsymbol{v}_{boat}$ and the static pressure field $p(\boldsymbol{r})$.

b) Show that the horizontal derivatives of the pressure field are given by

$$\frac{\partial p}{\partial x} = 0.00175 \frac{\mathrm{Pa}}{\mathrm{m}}, \quad \frac{\partial p}{\partial y} = -0.00179 \frac{\mathrm{Pa}}{\mathrm{m}}.$$

c) We are at 43°N and the density of the air can be assumed to be 1.2 kg/m^3. Determine the geostrophic wind vector in the area. Make a sketch of the pressure field and the geostrophic wind.

17. *Isobar surfaces (6p)*
On a weather map, isobars are *lines*, representing constant pressure for fixed height (near ground). Adding the vertical dimension, we get *surfaces* of constant pressure.

a) [3p] For an incompressible fluid, show that constant pressure in a region with a *geostrophic flow* $\mathbf{v} = (v_0,\ 0,\ 0)$ is described by a surface $z = Cy + B$, for all x. Express C in v_0 and the physical constants appearing in the equation for geostrophic balance.

b) [1p] Show that the result in (a) does not require constant density, and in fact is valid under geostrophic balance, independently of any particular relation between pressure p and density ρ.

c) [2p] An airplane flies above Lund at height 980 m (where we assume geostrophic balance to hold). It measures strict western wind of 40 m/s (western wind means that $\mathbf{v}$ points east), and a pressure p_L. Later, the airplane is near Berlin, 350 km straight south of Lund. Assuming constant wind during the whole trip (we do not worry too much about realism here. . .), at what altitude would the airplane measure the same pressure? Make rough numerical estimates when needed.

18. *Missing data (6p)*
A weather station (or balloon, or ship) measures the wind to be 6 m/s, pointing 30° west of the north direction. The pressure is measured to be 100.5 kPa. From another position, 20 km south of the first, the wind is the same but pressure data is missing.

a) [2p] Assuming geostrophic balance and a constant pressure gradient between the measurment positions, find an expression for the missing pressure data. Define all variables introduced in the expression. (Reminder: $\sin(30°) = 1/2$, $\cos(30°) = \sqrt{3}/2$.)

b) [1p] If the positions are in the southern hemisphere, would the result for the missing pressure data be higher or lower than 100.5 kPa?

c) [3p] Assume that the wind conditions are fairly similar over distances of about 500 km (remarkably steady weather, this is). Discuss in what regions on our planet that our approach to find the missing pressure would be misleading! The answer need not be very precise (the word "misleading" is not well defined), but should be motivated with estimates and calculations. You may set the earth's angular velocity to $0.5 \cdot 10^{-4}\,\mathrm{s}^{-1}$ and assume a constant air density $1.2\,\mathrm{kg/m^3}$.

19. *Book 18.2 with hints*
Some needed constants are given in example 18.1 on page314. Let the z direction be vertical and the flow in the x direction. Then pressure $p(y, z)$ will depend on an equation like (18.15).

Introduce a function $f(y)$ that describes the height of the interface at position y. The pressure at the interface is then $p_{\mathrm{i}}(y) = p(y, z = f(y))$. Calculate $\frac{\mathrm{d}p_{\mathrm{i}}(y)}{\mathrm{d}y} = \frac{\partial p}{\partial y} + f'(y)\frac{\partial p}{\partial z}$. Now comes the trick: since you are at the interface, this equation should give the same result if you use ρ and U from the brackish or saline region, respectively. Use the trick to determine $f'(y)$ and then the height difference between the west and east bank.

20. *Extra-curricular: Check section 18.5*
For your own enjoyment, check the numerical estimates and the results in section 18.5.

Chapter 13

21. ***Book 13.2, 13.3, 13.4***

22. ***Ideal flows (6p)***
Below are three two-dimensional velocity fields given in non-dimensional variables.

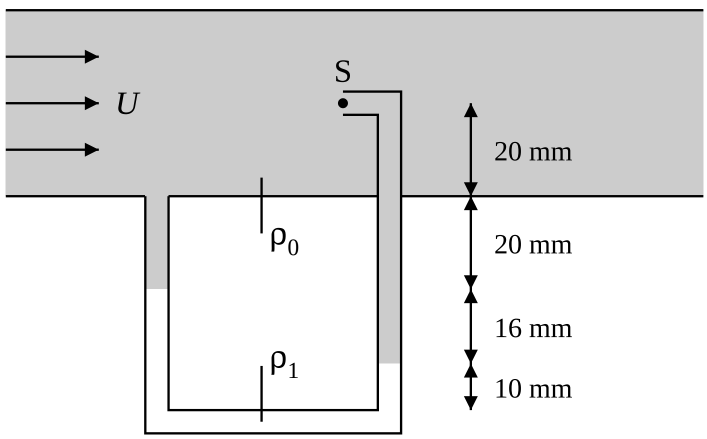
$$\begin{aligned} i) \quad & \mathbf{v} = (t\sin x,\ 0) \\ ii) \quad & \mathbf{v} = (y+t,\ 2) \\ iii) \quad & \mathbf{v} = (x,\ -y) \end{aligned}$$

a) [1p] For each one of them, determine if it represents steady flow or not.

b) [2p] For each one of them, determine if it represents irrotational flow or not.

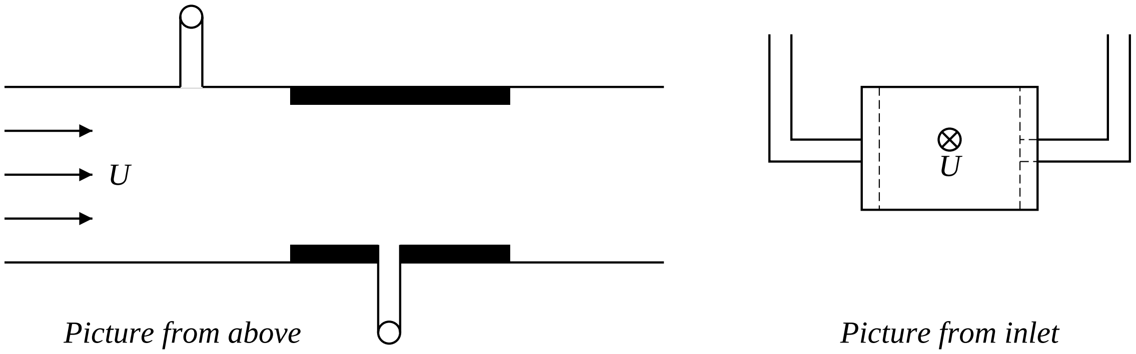
c) [3p] Could any of them be the velocity field for an incompressible ideal flow in the domain $y > 0$, bounded from below by a stationary wall at $y = 0$? If so, compute the corresponding pressure at zero gravity, with the constant non-dimensional density $\rho_0 = 2$.

23. ***Pitot by-pass (6p)***



The figure above illustrates a steady water flow in a pipe. A small tube is connected at the bottom of the pipe, and returns into the pipe as a Pitot tube. The Pitot tube is directed towards the flow and the point S at the entrance is a stagnation point where the fluid velocity is 0. The lower part of the by-pass is filled with mercury, while the rest is filled with water. The fluids in the by-pass are assumed to be completely still. At the entrance to the pipe, the flow is assumed to be uniform with speed U. Determine U by using the information in the figure and making reasonable assumptions. As approximations, you may use the gravitational acceleration $g \approx 10\,\mathrm{m/s}$, the water density $\rho_0 \approx 1000\,\mathrm{kg/m^3}$ and the mercury density $\rho_1 \approx 13.5\rho_0$.

24. ***Velocity Measurement (6p)***



Consider steady water flow filling a rectangular pipe with constant height. The width of the pipe is mostly constant, but in one region the inner of the pipe is narrowed to 80% of the width. The water velocity in the wide region is $1\,\mathrm{m/s}$.

Two side tubes are attached on the walls, one in the narrow region and one in a wide region. The tubes bend upwards and water entering from the pipe reaches different heights in the two tubes. The tubes are open on top and in contact with air.

a) [1p] Making reasonable assumptions, which of the two side tubes has the higher water surface?

b) [5p] What is the height difference? As approximations, you may use the gravitational acceleration $\approx 10\,\mathrm{m/s}$, water density $\approx 1000\,\mathrm{kg/m^3}$ and air pressure $\approx 100\,\mathrm{kPa}$.

Chapter 14

25. ***Book 14.1,14.3,14.5***

Chapter 25

26. ***Book 25.3***

27. ***How to measure a water wave***

It's not easy to determine the shape of a wave at sea. One possibility is to measure the pressure at the bottom of the ocean and try to recover the surface from these measurements. For this one needs a relation between the surface elevation and the pressure at the bottom. We assume that the water can be modelled as an ideal, incompressible and irrotational fluid with constant density ρ_0 and that the gravitational force is given by $\mathbf{g} = -g_0\hat{\mathbf{e}}_z$.

a) Assume that we can use a hydrostatic approximation of the pressure and that the bottom is flat and given by $z = -d$, while the surface is given by $z = h(x, y, t)$. Let $\Delta p = p|_{z=-d} - p_0$ be the pressure at the bottom minus the constant atmospheric pressure. Show that

$$h = \frac{\Delta p}{\rho_0 g_0} - d$$

b) One can improve the calculations for small-amplitude waves by using the linear theory for water waves. Assuming that the surface has the form $h = a\cos(kx - \omega t)$, one can then use the relations

$$\Psi = \frac{a\omega}{k}\frac{\cosh(k(z+d))}{\sinh(kd)}\sin(kx - \omega t),$$
$$p = p_0 - \rho_0 g_0\left(z - a\frac{\cosh(k(z+d))}{\cosh(kd)}\cos(kx - \omega t)\right),$$

where

$$\cosh(x) = \frac{e^x + e^{-x}}{2}.$$

Show that this gives

$$h = \left(\frac{\Delta p}{\rho_0 g_0} - d\right)\cosh(kd),$$

c) Let's now compare the results of the two methods. Assume that h has the exact form $h = a\cos(kx - \omega t)$, and that the linear theory gives a correct result for Δp. Show that the method to extract h from Δp in a) yields the surface

$$h = \frac{a}{\cosh(kd)}\cos(kx - \omega t).$$

d) How big is the the scale factor $\frac{1}{\cosh(kd)}$ for a tsunami with wavelength approximately 100 km on water of depth 4 km? How big is it for a wave of wavelength 2 m and depth 10 m? What's the conclusion? *Hint*: One can estimate $\cosh(x) \approx 1 + \frac{1}{2}x^2$ for $|x| < 1$ and $\cosh(x) \sim \frac{1}{2}10^{\frac{3}{7}|x|}$ for $|x| \gg 1$.

28. *Deep-water Waves (7p)*

Consider water with constant density ρ_0 and in constant gravity $\boldsymbol{g} = (0,\, 0,\, -g_0)$. The water is at rest with its surface at $z = 0$ amd a large depth d. We add a perturbation which creates a small amplitude surface wave, so that the surface is at

$$h(x,t) = a\cos(kx - \omega t).$$

If a is small enough, we can neglect the advective term in Euler's equations, so that the velocity is determined by

$$\frac{\partial \boldsymbol{v}}{\partial t} = \boldsymbol{g} - \frac{1}{\rho_0}\nabla p. \tag{1}$$

a) [2p] Show that constant density and eq. (1) implies that

$$\Phi^*(x,z,t) = g_0 z + \frac{1}{\rho_0}p(x,z,t)$$

satisfies the Laplace equation, $\nabla^2\Phi^* = 0$.

b) [3p] Show that

$$\Phi^* = \frac{a\omega^2}{k}\exp(kz)\cos(kx - \omega t) + C \tag{2}$$

is a solution. Find $\frac{\partial v_z}{\partial t}$ (or if you prefer, v_z) and show that it approximately satisfies the boundary conditions for $z \to -\infty$ and $z = h$. What are the constraints needed on the amplitude a for the approximations at $z = h$ to be valid?

c) [2p] Assume constant pressure p_0 at the surface and show, under the same approximations, the dispersion relation

$$\omega^2 = g_0 k.$$

Chapter 15

29. *Book 15.1, 15.4*

30. *Frictionless circulation*

In a Newtonian fluid, the stress tensor $\boldsymbol{\sigma}$ is linear in the components of the "velocity gradient tensor" $\boldsymbol{V}$, where $V_{ij} = \frac{\partial v_j}{\partial x_i}$. The most general 2-rank tensor for an isotropic fluid is then

$$\boldsymbol{\sigma} = -p\mathbb{1} + \eta_a \boldsymbol{V} + \eta_b \boldsymbol{V}^\top + \widehat{\eta}\mathbb{1}(\nabla \cdot \boldsymbol{v}),$$

where p is pressure, $\mathbb{1}$ is the identity matrix, $\boldsymbol{V}^\top$ is the transpose of $\boldsymbol{V}$.

Consider steady rotating flow with constant angular velocity $\boldsymbol{\omega} = \omega\widehat{\mathbf{e}}_3$, so that $\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{\omega} \times \boldsymbol{x}$.

a) Calculate the velocity gradient tensor for this flow.

An important feature of this flow is that *there is no friction.* This implies that all the frictional terms in the stress tensor $\boldsymbol{\sigma}$ must vanish, so that $\boldsymbol{\sigma} = -p\mathbb{1}$.

b) Show that this implies $\eta_a = \eta_b$. In other words, that $\boldsymbol{\sigma}$ for a Newtonian fluid must be symmetric.

31. *A pipe*

Water flows in a pipe pointing in the x direction. The cross-section of the pipe is constant, and we look for a steady solution to the flow where the velocity is only along the pipe and does not change as it travels through the pipe. In other words,

$$\boldsymbol{v} = (\ v_x(y,z),\ 0,\ 0\ ).$$

a) Show that the co-moving derivate of the velocity is zero.

b) Assuming constant density ρ_0 for the water and a constant gravitational acceleration $-g_0$ along the z direction, write down what remains of Navier–Stokes equations, when vanishing terms have been removed.

c) Show that the presure must be of the form

$$p = q(x) - \rho_0 g_0 z.$$

The function q depends only on x and may at this point be undetermined.

d) Show that the pressure fall per length is constant along the x direction,

$$\frac{\partial p}{\partial x} = -\eta G,$$

where η is the (constant) viscosity of water, and G is an unknown constant of appropriate dimension.

e) The only variables that can determine G are different length scales for the pipe (radius, height, width, etc.) and the average velocity $\langle v_x \rangle$ of the flow. Use dimensional arguments to show that G must be proportional to $\langle v_x \rangle$.

32. *Shear wave*

Verify that v_x in eq. (15.12) solves eq. (15.5) and satisfies the stated boundary conditions.

33. *Sound attenuation*

Verify eqs. (15.36), (15.38) and (15.39). Note that eq. (15.37) defines ω_0, and so is not an equation to be verified.

34. *Book 15.6, revised and with hints*

The purpose of this problem is to show that many of the equations we see in the book can be used to solve problems with rather general boundary conditions. If we couldn't, what would the point of discussing the equations in the first place? The problem 15.6 is, however, quite mathematical and is here split in smaller parts with extra hints. Along the way, a misprint is corrected.

The function $f(s)$ obeys eq. (15.16),

$$f''(s) + \frac{1}{2} s f'(s) = 0$$

and satisfies the boundary values

$$f(0) = 1, \quad f(\infty) = 0.$$

(These two boundary values will be needed a bit here and there in the problem.)

As stated in the book, $f\left(\frac{y}{\sqrt{\nu t}}\right)$ solves

$$\frac{\partial f}{\partial t} = \nu \frac{\partial^2 f}{\partial y^2}. \tag{3}$$

a) Verify this.

b) Show that $f\left(\frac{y}{\sqrt{\nu(t-t')}}\right)$ also solves eq. (3).

Now we have gathered all we need to know about f! We do not need the expression for f, (eq 15.17). But it is nice to see that the function exists.

The $v_x(y,t)$ in the problem is of the form

$$v_x(y,t) = \int_0^t h(y,t,t')\mathrm{d}t'.$$

This implies

$$\frac{\partial v_x}{\partial t} = h(y,t,t) + \int_0^t \frac{\partial}{\partial t} h(y,t,t')\mathrm{d}t', \tag{4}$$

$$\frac{\partial^2 v_x}{\partial y^2} = \int_0^t \frac{\partial^2}{\partial y^2} h(y,t,t')\mathrm{d}t'. \tag{5}$$

To be consistent with the definition of f in the book, the integrand h must be

$$h(y,t,t') = f\left(\frac{y}{\sqrt{\nu(t-t')}}\right)\dot{U}(t'). \tag{6}$$

(In problem 15.6, f has been written as $1-f$.)

In h we have a factor $\dot{U}(t')$ which means $\frac{\mathrm{d}}{\mathrm{d}t'}U(t')$. Note that t and t' are two independent variables, so that $\frac{\partial}{\partial t}\dot{U}(t')=0$. The expression $h(y,t,t)$ means $h(y,t,t')$ where t' happens to have the same value as t.

c) Using the expression stated here for h, show that $v_x(y,t)$ solves eq. (15.5). (You do not need to do calculate any derivative of f explicitly. Instead, make use of the result in **(b)**.)

d) Show that we fulfil the boundary condition $v_x(y,0)=0$. (This is not a trick question, it is as simple as it seems.)

e) Show that $v_x(0,t)=U(t)-U(0)$. Since the problem states $U(0)=0$, we fulfil the other boundary condition $v_x(0,t)=U(t)$.

Chapter 18.3

35. *Ekman layer: Check equation 18.27*

a) Verify that (18.27) solves (18.23).

b) Verify that (18.27) satisfies the stated boundary conditions.

c) Verify the approximate direction of the wind near ground, $0<z\ll\delta$.

36. *Exam 2015-06-05, problem 5 "The Gulf Stream"*

37. *Exam 2016-06-03, problem 3 "Barotropic, Geostrophic Catastrophe"*