Integrating gradients

1 dimension

The "gradient" of a function $f(x)$ in one dimension (i.e., depending on only one variable) is just the derivative, $f'(x)$. We want to solve

$$f'(x) = k(x), \tag{1}$$

where $k(x)$ is a known function. When we find a primitive function $K(x)$ to $k(x)$, the general form of f is K plus an arbitrary constant,

$$f(x) = K(x) + C. \tag{2}$$

Example: With $f'(x) = 2/x$ we find $f(x) = 2\ln x + C$, where C is an undetermined constant.

2 dimensions

We let the function depend on two variables, $f(x, y)$. When the gradient ∇f is known, we have known functions k_1 and k_2 for the partial derivatives:

$$\begin{aligned} \frac{\partial}{\partial x} f(x,y) &= k_1(x,y), \\ \frac{\partial}{\partial y} f(x,y) &= k_2(x,y). \end{aligned} \tag{3}$$

To solve this, we integrate one of them. To be specific, we here integrate $\frac{\partial f}{\partial x}$ over x. We find a primitive function *with respect to* x (thus keeping y constant) to k_1 and call it k_3. The general form of f will be k_3 plus an arbitrary term whose x-derivative is zero. In other words,

$$f(x, y) = k_3(x, y) + B(y), \tag{4}$$

where $B(y)$ is an unknown function of y. We have made some progress, because we have replaced an unknown function of two variables with another unknown function depending only on one variable.

The best way to come further is *not* to integrate $\frac{\partial f}{\partial y}$ over y. That would give us a second unknown function, $D(x)$. Instead, we find B via B' as follows:

$$B'(y) = \frac{\partial}{\partial y}\left[f(x,y) - k_3(x,y)\right] = k_2(x,y) - \frac{\partial}{\partial y} k_3(x,y). \tag{5}$$

Since k_3 is known, so is $\frac{\partial}{\partial y} k_3$, and the whole right hand side is known. Here, we call it k_4. We are back to the familiar one-dimensional case

$$B'(y) = k_4(y), \tag{6}$$

with solution $B(y) = K_4(y) + C$. This is plugged into eq. (4) to find f.

Example: We want to find f satisfying $\frac{\partial}{\partial x} f = 2y/x$ and $\frac{\partial}{\partial y} f = 2(\ln x + y)$. We integrate $\frac{\partial}{\partial x} f$ over x to get $f = 2y\ln x + B(y)$. The derivative of B becomes $B'(y) = \frac{\partial}{\partial y} f - \frac{\partial}{\partial y} 2y \ln x = 2\ln x + 2y - 2\ln x = 2y$. This has solution $B(y) = y^2 + C$. Our function f is $f = 2y\ln x + y^2 + C$.

Same example: This time, we start to integrate $\frac{\partial}{\partial y} f$ over y to get $f = 2y\ln x + y^2 + D(x)$. We find $D'(x) = \frac{\partial}{\partial x} f - \frac{\partial}{\partial x}\left(2y\ln x + y^2\right) = 2y/x - 2y/x = 0$. The solution to $D'(x) = 0$ is $D(x) = C$, so again we get the answer $f(x, y) = 2y\ln x + y^2 + C$.

Important control

Since $B(y)$ is independent of x, so is $B'(y)$. That implies that the known expression k_4 in eq. (6) must be independent of x,

$$k_2(x, y) - \frac{\partial}{\partial y} k_3(x, y) = k_4(y). \tag{7}$$

So, the x-dependece in k_2 and $\frac{\partial}{\partial y} k_3$ *must* cancel.

What if it doesn't?

It could be that the whole problem lacks a solution. Since $\frac{\partial^2}{\partial x \partial y} f = \frac{\partial^2}{\partial y \partial x} f$, we must have

$$\frac{\partial}{\partial y} k_1 = \frac{\partial}{\partial x} k_2. \tag{8}$$

If that is not the case, the vector field $\mathbf{k} = (k_1, k_2)$ cannot be expressed as the gradient of a scalar field, and f does not exist.

Example: The gradient $\frac{\partial}{\partial x} f = 2y/x$, $\frac{\partial}{\partial y} f = 2 \ln x + 2y$ has a solution, since $\frac{\partial}{\partial y}(2y/x) = 2/x$ is the same as $\frac{\partial}{\partial x}(2 \ln x + 2y) = 2/x$. As a counter-example, the equations $\frac{\partial}{\partial x} f = xy$ and $\frac{\partial}{\partial y} f = y^2$ lack a solution, since $\frac{\partial}{\partial y}(xy) = x$ differs from $\frac{\partial}{\partial x}(y^2) = 0$.

If eq. (8) is fulfilled and we still find ourselves with some x-dependence in k_4, we have made some simple mistake. We must go back and find it before we continue.

Example: We have accidentally used the derivative instead of primitive function when we integrated $2y/x$ over x, and got $f = -2y/x^2 + B(y)$. Continuing, we find $B'(y) = \frac{\partial}{\partial y} f - \frac{\partial}{\partial y}(-2y/x^2) = 2 \ln x + 2y + 2/x^2$. We cannot cancel the x-dependence, and we get warned that we have erred.

3 dimensions

The approach is again to integrate one equation and reduce the problem to one with a lower dimensionality. Starting with known expressions k_i in

$$\begin{aligned} \frac{\partial}{\partial x} f(x, y, z) &= k_1(x, y, z), \\ \frac{\partial}{\partial y} f(x, y, z) &= k_2(x, y, z), \\ \frac{\partial}{\partial z} f(x, y, z) &= k_3(x, y, z), \end{aligned} \tag{9}$$

we integrate one line of our choice. Here, we pick $\frac{\partial}{\partial x} f$, integrate over x and get

$$f(x, y, z) = k_4(x, y, z) + A(y, z), \tag{10}$$

where k_4 is a known expression satisfying $\frac{\partial}{\partial x} k_4 = k_1$. We then eliminate f by taking other partial derivatives on A:

$$\begin{aligned} \frac{\partial}{\partial y} A(y, z) &= \frac{\partial}{\partial y} [f(x, y, z) - k_4(x, y, z)] = k_5(y, z), \\ \frac{\partial}{\partial z} A(y, z) &= \frac{\partial}{\partial z} [f(x, y, z) - k_4(x, y, z)] = k_6(y, z). \end{aligned} \tag{11}$$

We are back to the 2-dimensional case, with known expressions for both $\frac{\partial}{\partial y} A$ and $\frac{\partial}{\partial z} A$. This we know how to solve. Once A is found, eq. (10) determines f.

The same important control

Again, there cannot be any x dependence in the known expressions for $\frac{\partial}{\partial y}A$ and $\frac{\partial}{\partial z}A$. If there is, we can control the existence of a solution by looking at cross derivatives, to see if $\frac{\partial}{\partial y}k_1 = \frac{\partial}{\partial x}k_2$, $\frac{\partial}{\partial z}k_1 = \frac{\partial}{\partial x}k_3$, and $\frac{\partial}{\partial z}k_2 = \frac{\partial}{\partial y}k_3$. In other words, we look at the rotation of $\mathbf{k} = (k_1, k_2, k_3)$. If $\nabla \times \mathbf{k} \neq 0$, then $\mathbf{k}$ cannot be the gradient of a scalar field.

When we work with a problem where $\mathbf{k} = \nabla f$ is possible, we may still find ourselves with some x-dependence left where it should not be. Then we have made some mistake and must go back and find it before we continue.

Example: Problem "Find the Pressure" on pdf with recommended problems (under appendix C).