These solutions are a bit verbose, since many students appreciated a thorough discussion on “how to think” when the exam results were orally presented.

1. The Standard Atmosphere (6p)
The temperature of the atmosphere at height \( z \) above ground is roughly

\[
T(z) = T_0 - az,
\]

where \( T_0 \) is the ground temperature and \( a \) is a parameter. With \( a = 6.5 \text{ K/km} \), this parametrization works well for the lowest 11 km of the atmosphere. The air is assumed to obey the ideal gas law, which can be written

\[
p = \rho R_{\text{air}} T.
\]

The value of \( R_{\text{air}} \) (the “specific gas constant” for air) is \( 287 \text{ m}^2/(\text{s}^2 \text{ K}) \).

\textbf{a)} [3p] Assume hydrostatic equilibrium and constant gravitational acceleration \( g_0 \approx 10 \text{ m/s}^2 \). Let \( p_0 \) and \( \rho_0 \) be the pressure and density at ground, respectively. Show that \( p(z) \) is of the form

\[
p(z) = p_0(1 - bz)^c
\]

and find expressions for \( b \) and \( c \).

\textit{Solution:} To solve this kind of problem, just have “a glance” at what is asked for, and then focus on the information given. Think of as a “balance game” between unknown variables and (independent) equations.

\textbf{Asked for:} a relation between \( p \) and \( z \).

\textbf{Given:} an equation relating \( T \) and \( z \), and one relating \( p, \rho \) and \( z \).

We have two equations with four unknowns. That can be reduced to a single equation with three unknowns, but not with just two.

But the problem also states “hydrostatic equilibrium” at constant gravity. That is the real course knowledge that enters the problem. We know it means

\[
\frac{\partial p}{\partial z} = -g_0\rho.
\]

This is almost a relation between \( p, \rho \) and \( z \), though the differential \( p'(z) \) makes it a bit more tricky.
Now, we have three equations with four unknowns and it should be possible to reduce it to one relation with two unknowns.

Quite a few students now assumed constant density and readily solved eq. (4). However, then you do not need the information in eqs. (1) and (2), which should make you suspicious. With three equations and four unknowns, there is no need to assume constant density. Furthermore if the density would be constant, then we would have three independent equations with only three unknowns, which makes the problem so called “over-determined”. It could of course be that our three equations are not independent (meaning that one of them can be derived from the other two) but before we know that to be the case, we cannot assume it.

Therefore, we do not assume constant density and we do not look for any other miraculous short-cuts. The balance game is done, and it is time to do the math work.

To make use of eq. (4) we need to integrate it, and then we don’t want any other unknowns than $p$ and $z$ (since there is a $p'(z)$). Luckily, the answer also wants us to focus on $p$ and $z$. Thus, we start to eliminate $T$ and $\rho$.

We use eq. (1) to eliminate $T$ from eq. (2) and get

$$p = \rho R_{\text{air}}(T_0 - az).$$

(5)

We use this to eliminate $\rho$ from eq. (4) and get

$$\frac{\partial p}{\partial z} = -g_0 \frac{p}{R_{\text{air}}(T_0 - az)}.$$

(6)

Advice: Avoid cluttered notation. It is a matter of taste, but now there are so many constants that we may start making "unforced errors" with them. I would introduce

$$N = \frac{g_0}{R_{\text{air}}T_0}$$

(7)

and

$$\alpha = a/T_0.$$

(8)

Next we need to recall how to solve the differential equation. We note that our equation allows separation of variables, which is always a good idea. We get

$$\frac{dp}{p} = -\frac{N}{1 - \alpha z} dz.$$

(9)

which yealds

$$\ln p = \frac{N}{\alpha} \ln(1 - \alpha z) + C,$$

(10)

where $C$ is the integration constant. Before we move one we check ourselves by differentiating our result and confirm that we retrieve eq. (9)!. If it is an easy integral, we do it in the head, but we do it.
Since we hunt $p$, we exponentiate to get

$$p = e^C(1 - \alpha z)^{N/\alpha}. \quad (11)$$

To get there, we need to use some standard manipulations, like $k \ln x = \ln x^k$ and $e^{a+b} = e^a e^b$. We do so because we glance at the answer eq. (3) and note that we are coming close.

From the problem formulation, eq. (3), we see that we are allowed to work with a constant $p_0$, and that it is determined by $p(z = 0) = p_0$. With $z = 0$ we get

$$p = p_0(1 - \alpha z)^{N/\alpha}. \quad (12)$$

This is the desired form, with $b = \alpha$ and $c = N/\alpha$. However, since $\alpha$ and $N$ were constants introduced by us during the work, we try to return to the original constants. We find $b = a/T_0$ and $c = g_0/(R_{\text{air}} a)$.

**We are not done!** Now, we check dimensions.

In the expression $1 - bz$, the term $bz$ must be dimensionless. From eq. (1) and the value of $a$ given right thereafter, we see that the dimension of $a$ is temperature/length. Thus, the dimension of $b = a/T_0$ is 1 over length, and $bz$ is dimensionless. Fine.

Furthermore, the exponent $c = g_0/(R_{\text{air}} a)$ must be dimensionless (you cannot raise something to the power of a length, or a pressure, or anything with dimension). Check that $c$ is dimensionless in your favourite way, but do it!

Actually, a good point to check dimensions is right after eq. (9). Both $N$ and $\alpha$ should be inverse lengths. If something is wrong there, one looses precious exam time if one soldiers on with calculations. But we also check dimensions in the final answer, because it is quite possible to make mistakes between eq. (9) and the final result.

**b) [1p]** Show that the atmospheric model is polytropic, $p = C \rho^\gamma$, where $C$ and $\gamma$ are constants. Find an expression for $\gamma$.

**Solution:** This time we are asked for a single relation between two other unknowns, $p$ and $\rho$. The balance game is the same as before. After our work with the previous problem, we now have many equations, but we know that they are not all independent. The three original ones are a safe starting point, but it is a good idea to make some use of our previous work, in particular to get rid of the differential equation.

Our result in eq. (3) relates $p$ and $z$, and eq. (5) relates $p$, $\rho$ and $z$. We want to eliminate $z$. That looks tough, since $z$ sits deep inside a parenthesis, but we note that it is the same parenthesis in both relations, since $(T_0 - az)$ in eq. (5) can be written $T_0(1 - bz)$. Thus, we can use e.g. eq. (3) to eliminate $1 - bz$ from eq. (5) and get

$$p = \rho R_{\text{air}} T_0 \left( \frac{p}{p_0} \right)^{1/c}. \quad (13)$$
Roughly now is a good time to introduce a prettier notation with new constants, and this
time we can be a bit sloppy, because the problem does not ask us to determine the constant
in front of $\rho^\gamma$. We have $p = p^{1/c} \rho C$. Separating gives $p^{(c-1)/c} = C \rho$. (As a side note referring
to the constant density mistake many did in problem a: if we would have $c = 1$ we would
indeed find $\rho$ to be constant, but until now, we cannot assume it. And in our case, putting
in numbers we find $c \neq 1$.) We solve for $p$ to get $p = C' \rho^{c/(c-1)}$. We have shown what we
should and found $\gamma = c/(c - 1)$. We do not need to re-phrase the constant $C'$ in original
parameters, because it is not asked for. Phew!

c) [2p] Suppose the ground temperature is $T_0 = 286$ K ($\approx 13^\circ$C). Make a rough numerical
estimate of the density at height 11 km, relative to ground density – in other words, estimate
the ratio $\rho(11 \text{ km})/\rho_0$. Hint: 286/6.5=44. You may use the estimate $g_0/R_{\text{air}}a \approx 5$.

Solution: When we are asked for $\rho(11 \text{ km})/\rho_0$ we should see if we can find the general so-
lution $\rho(z)$, and then particularly calculate it for $z = 0$ and $z = z_1 = 11$ km. Now, $\rho(z)$ is
one relation with two unknows ($\rho$ and $z$). The balance game is as before and we can reuse
a lot. Using e.g. eq. (5) to eliminate $p$ from eq. (3) gives $(1 - bz)^c = \rho R_{\text{air}}a(1 - bz)$ and
$\rho = \frac{1}{aR_{\text{air}}}(1 - bz)^{c-1} = \rho_0(1 - bz)^{c-1}$. In the last step, we renamed the constant to satisfy the
boundary condition $\rho(0) = \rho_0$.

The numerical help in the problem gives $c \approx 5$ and $b \approx 1/(44 \text{ km})$ so that $bz_1 \approx 1/4$ and
$\rho(z_1)/\rho_0 \approx (1 - \frac{1}{4})^4 = (3/4)^4 \sim 0.3$. 
2. Pressure in a tornado (10p)
The “Rankine vortex” can be used as a simple model for tornadoes. It’s defined by
\[ \mathbf{v} = (-\Omega y, \Omega x, 0), \]
where the angular frequency \( \Omega \) depends on the horizontal radius \( r = \sqrt{x^2 + y^2} \) as
\[
\Omega(r) = \begin{cases} 
\Omega_0, & 0 \leq r \leq R, \\
\Omega_0 \frac{R^2}{r^2}, & r > R.
\end{cases}
\]
\( R > 0 \) and \( \Omega_0 \) are constants. The region \( 0 \leq r < R \) is called the core of the vortex.

First, we look in the core,
\[ \mathbf{v}(\text{core}) = (-\Omega_0 y, \Omega_0 x, 0). \]

\textbf{a)} [1p] Show that the vorticity in the core is a non-zero constant.
\textit{Solution:} The vorticity is \( \nabla \times \mathbf{v}(\text{core}) = (0, 0, 2\Omega_0). \)

\textbf{b)} [1p] Calculate the stream function in the core and show that the stream lines are circles.
\textit{Solution:} The stream function \( \psi \) satisfies \( \frac{\partial \psi}{\partial x} = v_y \) and \( \frac{\partial \psi}{\partial y} = -v_x \). Integrating the first gives \( \psi(x, y) = \frac{1}{2} \Omega_0 x^2 + A(y) \), so that \( \frac{\partial \psi}{\partial y} = A'(y) \). In other words \( A'(y) = -v_x \Rightarrow A = \frac{1}{2} \Omega_0 y^2 + B \), where \( B \) is a constant that we may put to zero (since the stream function is only interesting up to a constant). Putting together, we get \( \psi = \frac{1}{2} \Omega_0 (x^2 + y^2) \). We see that \( \psi \) is constant along circles, and since it is also constant along stream lines, they must be circles.

\textbf{c)} [2p] Assume that \( \mathbf{v}(\text{core}) \) describes the flow of an ideal, incompressible fluid with constant density \( \rho_0 \) and zero gravity. Calculate the pressure in the core. The answer should contain one free constant.
\textit{Solution:} The x-component of Euler’s equations gives \( \frac{\partial p}{\partial x} = -\rho_0(\mathbf{v} \cdot \nabla)v_x = \rho_0 \Omega_0 (v_x \frac{\partial \psi}{\partial x} + v_y \frac{\partial \psi}{\partial y}) = \rho_0 \Omega_0 v_y = \rho_0 \Omega_0^2 x. \) Integrating gives \( p = \frac{1}{2} \rho_0 \Omega_0^2 x^2 + A(y) \). We have introduced an unknown function \( A(y) \) but we know that \( \frac{\partial p}{\partial y} = \frac{\partial}{\partial y} \left( \frac{1}{2} \rho_0 \Omega_0^2 x^2 + A(y) \right) = A'(y) \). Together with the y-component of Euler’s equations, we find \( A'(y) = \rho_0 \Omega_0^2 y. \) Integrating and putting together gives the answer \( p = \frac{1}{2} \rho_0 \Omega_0^2 x^2 + B. \)

Next, we look outside the core, where the velocity is
\[ \mathbf{v}(\text{out}) = R^2 \left( -\frac{\Omega_0 y}{x^2 + y^2}, \frac{\Omega_0 x}{x^2 + y^2}, 0 \right) = \frac{R^2}{x^2 + y^2} (-\Omega_0 y, \Omega_0 x, 0). \]

\textbf{d)} [1p] Show that the vorticity outside the core is zero.
Solution: In planar flow, only $\omega_z$ can be nonzero. In our case $\omega_z = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} - \frac{\partial z}{\partial x} \cdot \frac{x}{x^2 + y^2} + \frac{\partial z}{\partial y} \cdot \frac{y}{x^2 + y^2}$. From $\frac{\partial}{\partial x} \frac{x}{x^2 + y^2} = \frac{1}{x^2 + y^2} - \frac{2x^2}{(x^2 + y^2)^2}$ and similarly for $y$, we get $\omega_z = \frac{2}{x^2 + y^2} - \frac{2(x^2 + y^2)}{(x^2 + y^2)^2} = 0$.

(The problem is much easier solved in cylindrical coordinates, but then you have to be comfortable with $\nabla$ in such a representation.)

e) [1p] Calculate the stream function outside the core and show that the stream lines are circles.

Solution: We recognize $x$ as (1/2 times) the inner derivative of $x^2$, so $\frac{\partial \psi}{\partial x} = v_y$ gives $\psi = \frac{R^2 \Omega}{2} \ln(x^2 + y^2) + A(y)$. From $v_x = -\frac{\partial \psi}{\partial y} = -\frac{\partial}{\partial y} \left( \frac{R^2 \Omega}{2} \ln(x^2 + y^2) \right) - A'(y) = -\frac{R^2}{x^2 + y^2} \Omega_0 y - A'(y)$ we get $A'(y) = 0$ and $A$ is a constant that we may set to zero (in this case). Again, $\psi$ is a function of $x^2 + y^2$ and so constant on circles.

f) [1p] For future convenience, show that

$$\mathbf{v}^{(\text{out})} \cdot \nabla \frac{R^2}{r^2} = 0.$$  

(This result is useful together with the product rule for derivatives. Obviously, the result may be used below even if you failed to show it.)

Solution: You can immediately use that $\nabla f$ is perpendicular to contour lines for $f$. In this case $f = f(r)$ so $\nabla f$ is perpendicular to circles. Then, the scalar product with circular $\mathbf{v}^{(\text{out})}$ follows. You can also do the math: $\frac{\partial}{\partial x} \frac{1}{x^2 + y^2} = -\frac{2x}{(x^2 + y^2)^2} = g(r)x$, so $\nabla R^2 / r^2 = R^2 g(r)(x, y, 0)$ and $\mathbf{v}^{(\text{out})} = h(r)(-y, x, 0)$ gives a scalar product proportional to $x(-y) + yx = 0$.

g) [2p] Calculate the pressure outside the core (under the same assumptions as in the core). The answer should contain one free constant.

Solution: We can attack this as we did in (c), and use (f) to find the advective term $(\mathbf{v} \cdot \nabla)\mathbf{v}$ without too much work, but better is to use the fact that the flow is irrotational, so that the Bernouilli field is constant. We get $H = \frac{1}{2} v^2 + \Phi + p/\rho_0$. Since we have zero gravity, we get $p = -\frac{1}{2} \rho_0 \Omega^2 \frac{R^4}{r^2} + C$, where $C$ is a constant.

Finally, we combine:

h) [1p] Assuming that the pressure is continuous at $r = R$, show that the difference $\Delta p$ in pressure between $r = 0$ and $r = \infty$ is $-\rho_0 \Omega^2 R^2$.

Comment: this explains the very low pressure in the centre of a tornado!

Solution: Continuous $p$ gives $\frac{1}{2} \rho_0 \Omega^2 R^2 + B = -\frac{1}{2} \rho_0 \Omega^2 R^2 + C$, or $C - B = \rho_0 \Omega^2 R^2$. Together with $p(0) = 0 + B$ and $p(\infty) = 0 + C$ we get the answer.

3. Half full or half empty (8p)
A large, open tank with constant cross-section $A_0$ and height $h$ is filled with water. At time $t = 0$, an outlet with cross-section $A < A_0$ is opened at the bottom, and water exits the outlet with a velocity $v$, while the surface in the tank sinks with a velocity $v_0$.

**a)** [1p] Assuming constant density, find a relation between $v$ and $v_0$, independent of $h$.

*Solution:* Constant density allows for Leonardo’s law, so we get $Av = A_0v_0$.

**b)** [1p] Adding the assumptions of steady, ideal flow and constant gravity, find a relation between $v$, $v_0$ and $h$.

*Solution:* Following a streamline from the surface to the outlet, we get $H = \frac{1}{2}v^2 + \frac{p_0}{\rho_0} + gh$ at the surface and $H = \frac{1}{2}v_0^2 + \frac{p_0}{\rho_0}$ at the outlet. (Both the surface and the outlet is in contact with air, which has pressure $p_0$.) Since $H$ is constant along streamlines, we find $\frac{1}{2}v^2 + gh = \frac{1}{2}v_0^2$.

**c)** [1p] Combining the above, express the velocity through the outlet as a function of $h$.

*Solution:* We eliminate $v_0$ to get $A\frac{v^2}{A_0} + 2gh = v_0^2$ and $v = \sqrt{2gh/\left(1 - A^2/A_0^2\right)}$.

**d)** [3p] Calculate the ratio $T_{1/2}/T$, where $T_{1/2}$ is the time required for the tank to empty halfway and $T$ the time required for the tank to empty completely.

*Solution:* The height $z(t)$ satisfies the initial condition $z(0) = h$ and $\dot{z} = -v_0 = -\sqrt{\frac{2ghz}{A_0^2/A^2 - 1}} = -B\sqrt{z}$. Separation of variables gives $\frac{dz}{\sqrt{z}} = -Bdt$ and $2\sqrt{z} + C = -Bt$. The initial condition determines $C$ and gives $2(\sqrt{z} - \sqrt{h}) = -Bt$. We get $t(z) \propto \frac{\sqrt{h} - \sqrt{z}}{\sqrt{h} - 1}$. The proportionality constant cancels in the ratio $T_{1/2}/T = t(h/2)/t(0) = (\sqrt{h} - \sqrt{h/2})/\sqrt{h} = 1 - \sqrt{1/2}$.

**e)** [2p] Hopefully, your result in (c) is un-physical in the limit $A \to A_0$ (representing a situation where almost the whole bottom of the tank is opened as an outlet). What assumption(s) above would be un-reasonable for large $A$?

*Solution:* In the extreme case that $A = A_0$, the bottom of the tank suddenly disappears. Then the fluid would fall freely, and accelerate. Therefore, steady flow is unreasonable. (When the velocity becomes very large, a time into free fall, the friction against tank walls would be more noticeable. Eventually, you could get steady flow where viscosity has to be included, but all our experiences of turning buckets upside-down tells us that the water has left any realistic tank long before.)

4. **Missing data** (6p)

A weather station (or balloon, or ship) measures the wind to be 6 m/s, pointing 30° west of the north direction. The pressure is measured to be 100.5 kPa. From another position, 20 km south of the first, the wind is the same but pressure data is missing.

**a)** [2p] Assuming geostrophic balance and a constant pressure gradient between the measurement positions, find an expression for the missing pressure data. Define all variables
introduced in the expression. (Reminder: \( \sin(30^\circ) = 1/2, \cos(30^\circ) = \sqrt{3}/2 \).)

**Solution:** Geostrophic balance implies \( \nabla p = -\rho 2 \Omega_z \times v \), where \( \rho \) is density and \( \Omega_z \) is the local (vertical) angular velocity of the earth. With the x-direction pointing north and the y-direction west, we get \( \frac{\partial p}{\partial x} = -\rho 2 (\Omega_y v_x - \Omega_z v_y) = \rho 2 \Omega_y v_y = \rho \Omega_z v_0 \), where \( v_y = v_0 \sin(30^\circ) = v_0/2 \) and \( v_0 = 6 \text{ m/s} \).

The two measurements are made at the same \( y \), but different \( x \). Constant pressure gradient means that the pressure difference between points \( A \) and \( B \) becomes \( p_A - p_B = (x_A - x_B) \frac{\partial p}{\partial x} + (y_A - y_B) \frac{\partial p}{\partial y} = (x_A - x_B) \rho \Omega_z v_0. \)

With \( A \) the position of known pressure, we get \( x_A - x_B \equiv L = +20 \text{ km} \) (point \( A \) is north of \( B \)). The missing data would be \( p_B = p_A - L \rho \Omega_z v_0. \)

**b)** [1p] If the positions are in the southern hemisphere, would the result for the missing pressure data be higher or lower than 100.5 kPa?

**Solution:** In the southern hemisphere, \( \Omega_z \) is negative, so \( p_B > p_A. \)

**c)** [3p] Assume that the wind conditions are fairly similar over distances of about 500 km (remarkably steady weather, this is). Discuss in what regions on our planet that our approach to find the missing pressure would be misleading! The answer need not be very precise (the word “misleading” is not well defined), but should be motivated with estimates and calculations. You may set the earth’s angular velocity to \( 0.5 \cdot 10^{-4} \text{ s}^{-1} \) and assume a constant air density \( 1.2 \text{ kg/m}^3 \).

**Solution:** We are provided with a typical length scale \( L = 500 \text{ km} \) and our only available estimate of a typical velocity is \( v_0. \) The Rossby number at an angle \( \theta \) from a pole of the earth is \( v_0/(2L|\cos \theta|) \sim 6/(2 \cdot 5 \cdot 10^5 \cdot 0.5 \cdot 10^{-4} |\cos \theta|) = 6/(50 |\cos \theta|). \) “Near” the equator, meaning \( |\cos \theta| < 1/8 \) or so, the Rossby number is too large for geostrophic balance to be a valid assumption. (A more trivial remark is that point \( A \) must be at least 20 km away from the south pole for point \( B \) to exist... )