Solutions to exam, FYTA14, 2015-06-05

Allowed material: One a4 sheet with notes, writing material.

30 points total, 15 points to pass, 24 points for distinction.

The solutions are too brief for full score, to highlight the main arguments. Examples of required sketches are not shown.

1. Navier–Stokes Basics (5p)

a) [1p] Write down the Navier–Stokes equation in vector notation. Name the included fields and constants.

Solution:

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \frac{1}{\rho}\boldsymbol{f} - \frac{1}{\rho}\nabla p + \frac{1}{\rho}\eta\nabla^2\boldsymbol{v} + \frac{1}{\rho}\left(\zeta + \frac{1}{3}\eta\right)\nabla(\nabla \cdot \boldsymbol{v}).$$

Where $\boldsymbol{v}$ is velocity, ρ is density, p is pressure, $\boldsymbol{f}$ is external forces (which can be represented by gravitational acceleration $\boldsymbol{g} = \boldsymbol{f}/\rho$), η is kinematic viscosity and ζ is "bulk viscosity". Viscosity η can be represented by kinematic viscosity $\nu = \eta/\rho$.

b) [3p] Which terms vanish if the flow is:
...incompressible? *Solution:* The one with $\nabla \cdot \boldsymbol{v}$.
...steady? *Solution:* $\frac{\partial \boldsymbol{v}}{\partial t}$
...ideal? *Solution:* All terms with viscosity or bulk viscosity.

c) [1p] How would you modify the Navier–Stokes equation if it was expressed in a coordinate system rotating with a constant angular vector $\boldsymbol{\Omega}$ relative to an inertial frame?

Solution: I would add fictitios forces $-\rho 2\boldsymbol{\Omega} \times \boldsymbol{v} + \rho\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r})$, where $\boldsymbol{r}$ is the position vector.

2. Surface Gravity Waves (6p)

a) [1p] Make a sketch of a surface gravity wave and the relevant physical quantities that are important for its propagation.

Solution: The sketch should include a wavy surface, and mention variables that enter the mathematical description: wavelength, amplitude, depth (to average surface level) density, pressure, and gravitational acceleration. Wave width (supposed to be approximately infinite) may also be mentioned.

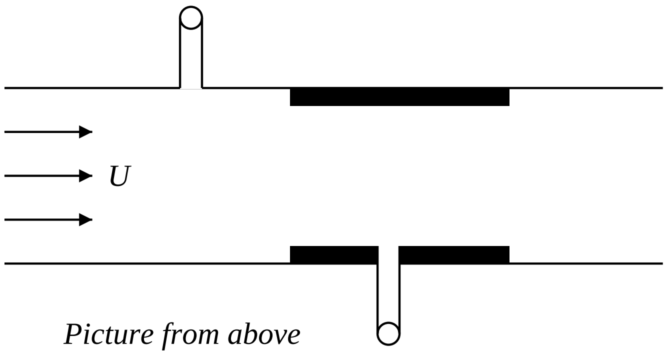
b) [2p] Give a qualitive description of what drives the propagation of a surface gravity wave.

Solution: The answer should clarify the interplay between gravity, pressure and incompressibility. Pressure will depend on the height of the surface, due to gravity. Pressure gradients will therefore accelerate water from under mounds towards valleys. Since water is (nearly) incompressible, the water near a mound surface sinks to fill the abandoned region below, and the water near valley surfaces is pushed upwards by incoming water below.

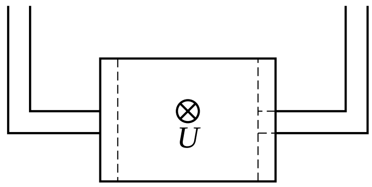
c) [3p] Describe the difference between the properties of a shallow-water wave and a deep-water wave.

Solution: Deep-water waves have wavelengths much smaller than the depth (depth is formally infinite). The celinity depends on wavelength, so that modes with long wavelengths travel faster. Shallow-water waves have wavelengths larger than depth. Celinity is then independent of wavelength, but depends on depth.

3. Velocity Measurement (6p)



Picture from above



Picture from inlet

Consider steady water flow filling a rectangular pipe with constant height. The width of the pipe is mostly constant, but in one region the inner of the pipe is narrowed to 80% of the width. The water velocity in the wide region is 1 m/s.

Two side tubes are attached on the walls, one in the narrow region and one in a wide region. The tubes bend upwards and water entering from the pipe reaches different heights in the two tubes. The tubes are open on top and in contact with air.

a) [1p] Making reasonable assumptions, which of the two side tubes has the higher water surface?

Solution Assuming ideal, incompressible flow, the velocity must be higher in the narrow region (Leonardo's law) which by Bernoulli's theorem lowers the pessure. The surface height is a measure of the pressure in the flow, so it will be higher in the pipe attached to the wide region.

b) [5p] What is the height difference? As approximations, you may use the gravitational acceleration $\approx 10\,\mathrm{m/s}$, water density $\approx 1000\,\mathrm{kg/m^3}$ and air pressure $\approx 100\,\mathrm{kPa}$.

Solution: In the side tubes we assume hydrostatic equilibrium. If the surface in tube i is at height h_i above its attachment to the pipe, the pressure p_i in the attachment point is $p_i = p_0 + \rho_0 g_0 h_i$. In the big pipe, ideal irrotational flow implies constant Bernoulli field $H = \frac{1}{2}v^2 + g_0 z + p/\rho_0$. With tubes attached at the same z, we find $\frac{1}{2}v_1^2 + p_1/\rho = \frac{1}{2}v_2^2 + p_2/\rho$ so that $h_1 - h_2 = \frac{1}{g_0\rho_0}(p_1 - p_2) = \frac{1}{2g_0}(v_2^2 - v_1^2)$. Finally, the equation of continuity in the form of Leonardo's law gives us $v_2 = v_1/0.8$, where $v_1 = 1\,\mathrm{m/s}$ is the velocity in the wide region. Putting all together gives $h_1 - h_2 = \frac{1}{20}\left(\frac{1}{0.8^2} - 1\right)$ m$= \frac{36}{1280}$ m$\sim 3\,\mathrm{cm}$.

4. Flow on a Plane (7p)

Consider steady water flow in the x-direction on a plane perpendicular to the z-direction, such that $\mathbf{v} = (v_x(z),\, 0,\, 0)$. The water surface is at constant height above the plane. Assume constant water density ρ_0 and a constant pressure p_0 at the surface.

Let the plane (and the coordinate system) be tilted with respect to the earth vertical, so that the gravitational acceleration is $\mathbf{g} = (g_x,\, 0,\, g_z)$, where g_x and g_z are constants.

a) [1p] Verify that the proposed velocity field is consistent with a constant density.

Solution: Constant density requires $\nabla \cdot \boldsymbol{v} = 0$. Indeed, we have $\nabla \cdot \boldsymbol{v} = \frac{\partial v_x}{\partial x} = 0$, since $v_x = v_x(z)$.

b) [2p] Show that the pressure in the fluid is independent of x. *Hint:* Use the z-component of the Navier–Stokes equations and the pressure boundary condition at the surface.

Solution: With steady flow and $\nabla \cdot \boldsymbol{v} = 0$, the z-component reads $0 = g_z - \frac{1}{\rho_0}\frac{\partial p}{\partial z}$. Integrating gives $p = \rho_0 g_z z + A(x, y)$, where A is an unknown function of x and y. However, the boundary condition at height $z = h$ (introducing h), is $p = p_0$, so that $A = p_0 - g_z \rho_0 h$ and independent of x (and y).

c) [1p] Knowing that the pressure is independent of x (even if you have not proven it), find an expression for $v_x''(z)$.

Solution: We look at the x-component of Navier–Stokes, where most terms become zero, so that it remains $0 = g_x + \nu\frac{\partial^2 v_x}{\partial z^2}$. Since v_x only depends on z, the partial derivative symbol is superfluous and we find $v_x''(z) = -g_x/\nu$.

d) [3p] Find an expression for $v_x(z)$ and determine all integration constants with the help of two boundary conditions:
1) The velocity at the plane
2) The shear forces at the surface. Assume that air imposes negligible shear forces on the water, so that the stress tensor component σ_{xz} is 0 at the water surface.

Solution: Integrating twice gives $v_x = -\frac{g_x}{2\nu}z^2 + C_1 z + C_2$, where C_1 and C_2 are constants. Defining coordinate system so that $z = 0$ at the plane gives $v_x = -\frac{g_x}{2\nu}z^2 + C_1 z$ from the no-slip boundary condition (making the reasonable assumption that the plane is still). The stress tensor component σ_{xy} is $\eta(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}) = \eta v_x'(z) = \eta(-\frac{g_x}{\nu} z + C_1)$. For this to vanish at the surface we must have $C_1 = \frac{g_x}{\nu} h$ so that $v_x = \frac{g_x}{\nu}\left(zh - \frac{1}{2}z^2\right)$. Since we have defined $z = h$ at the surface and $z = 0$ at the plane, the symbol h represents the height (perpendicular to the tilted plane) of the flow.

5. The Gulf Stream (6p)

The Gulf Stream runs counter-clockwise in the northern Atlantic Ocean (as a very simplified description). According to Wikipedia it is typically 100km wide and flows with a velocity of about 1 m/s. We assume that the current can be described by *geostrophic balance*, and that the water surface lies at constant pressure p_0.

a) [2p] Describe why there will be a height difference between the inner and outer edges of the current, and determine which edge that has the highest surface. Furthermore, assuming (boldly) that the width and speed of the flow is constant, where on earth would the height difference be maximized?

Solution: The Coriolis force in geostrophic balance will act to the right for streams in the northern hemisphere, which follows from the term $-2\Omega_z \widehat{\boldsymbol{e}}_z \times \boldsymbol{v}$. The pressure at the water surface is expected to be fairly constant, and a height difference will then create a pressure gradient which can componesate the Coriolis force. To do so, the surface must be higher in the right edge of the stream. For counter-clockwise motion, that is the outer edge. (This correponds to winds going counter-clockwise around a low pressure. The lower surface on the inner side creates a "low pressure" for water beneath the surface. The current goes counter-clockwise around that low pressure.)

b) [2p] Deeper down, the Gulf stream meets other water layers with different motion. At the lower regions of the Gulf stream there is therefore an Ekman layer where the flow disagrees with geostrophic balance. Qualitatively, what is the direction of the flow in the Ekman layer, and how does that affect the height difference between the outer and inner water surfaces?

Solution: The simplest argument is that viscosity will change the velocity to become smaller, and then the force from the pressure gradient will dominate. So there will be a velocity component in the direction of negative pressure gradient, inwards. This will reduce the height difference between the outer and inner edge.

Note that the question refers to an Ekman layer at the bottom of the stream, not a layer at the surface, where winds may start a water motion whose direction changes with depth due to Coriolis effects. However, at the surface, the flow is not influenced by any pressure gradient due to water height differences, so by-heart knowledge from one kind of layer does not apply to the other kind.

c) [1p] If the stream instead had run clock-wise, with preserved numerical values, would the height difference change in size or direction? Would the effects of the Ekman layer change? For the "yes" answers: what is the change?

Solution: The Coriolis effect still drives the flow to the right, so now the inner edge must be the higher (a clock-wise flow around a "high pressure" in the middle), with the same difference magnitude as before. The Ekman layer at the bottom will be influence by the new pressure gradient and go outwards. It will still reduce the height difference.

d) [1p] *A side-remark stated as a sub-problem*: When the counter-clockwise Gulf stream turns left, you may expect a pressure gradient to drive the change of velocity direction. With a typical speed U and radius R of the turn, the acceleration "left-wards" is U^2/R. With reasonable assumptions on R, show that this acceleration is negligible compared to other effects in geostrophic balance. Thus, this acceleration need not be considered in the rest of the problem!

Solution: We want to compare the "left-ward" (centripetal) acceleration the Coriolis acceleration effect, so we look at the Rossby number $U^2/R2\Omega U = U/2\Omega R$. A typical U was given as $1\,\mathrm{m/s}$ and the typical radius can hardly be smaller than the width of the stream, so a reasonable upper bound on the Rossby number $\sim 1/(10^{-4}10^5) = 0.1$. A student cleverly pointed out that these 10% cooperate with the Coriolis effect for counter-clockwise flow, but counter-acts it for clockwise flow, so the magnitude of the height difference in the two scenarios is not quite the same. A beautiful remark, which definitely is not needed for full score.