Solutions to Exam, FYTA14, 2016-06-03

Allowed material: One a4 sheet with notes, writing material.

30 points total, 15 points to pass, 24 points for distinction.

These solutions are in general too brief to give a full score. They are meant to help students reconstruct a good solution.

1. Deep-water Waves (7p)

Consider water with constant density ρ_0 and in constant gravity $\boldsymbol{g} = (0,\, 0,\, -g_0)$. The water is at rest with its surface at $z = 0$ amd a large depth d. We add a perturbation which creates a small amplitude surface wave, so that the surface is at

$$h(x,t) = a\cos(kx - \omega t).$$

If a is small enough, we can neglect the advective term in Euler's equations, so that the velocity is determined by

$$\frac{\partial \boldsymbol{v}}{\partial t} = \boldsymbol{g} - \frac{1}{\rho_0}\nabla p. \tag{1}$$

a) [2p] Show that constant density and eq. (1) implies that

$$\Phi^*(x,z,t) = g_0 z + \frac{1}{\rho_0} p(x,z,t)$$

satisfies the Laplace equation, $\nabla^2\Phi^* = 0$.

$\nabla^2\Phi^* = \nabla\cdot(\nabla\Phi^*) = -\nabla\cdot\frac{\partial \boldsymbol{v}}{\partial t} = -\frac{\partial}{\partial t}\nabla\cdot\boldsymbol{v} = 0$ since incompressibility implies $\nabla\cdot\boldsymbol{v} = 0$.

b) [3p] Show that

$$\Phi^* = \frac{a\omega^2}{k}\exp(kz)\cos(kx - \omega t) + C \tag{2}$$

is a solution. Find $\frac{\partial v_z}{\partial t}$ (or if you prefer, v_z) and show that it approximately satisfies the boundary conditions for $z \to -\infty$ and $z = h$. What are the constraints needed on the amplitude a for the approximations at $z = h$ to be valid?

That it is a solution follows from $\frac{\partial^2}{\partial x^2}\Phi^* = -k^2(\Phi^* - C)$, $\frac{\partial^2}{\partial y^2}\Phi^* = 0$, $\frac{\partial^2}{\partial z^2}{}^2 = k^2(\Psi^* - C)$. We get $\frac{\partial v_z}{\partial t} = -\frac{\partial}{\partial z}\Phi^* = -a\omega^2 \exp(kz)\cos(kx - \omega t)$. We do not expect any effets of the wave at large depths, so $\lim_{z\to-\infty} v_z = 0$. For this to be true at all times, we must have $\lim_{z\to-\infty} \frac{\partial}{\partial t} v_z = 0$, which is satisfied due to the $\exp(kz)$ factor.
At $z = h$ we want $\frac{\partial v_z}{\partial t} = \frac{\partial^2 h}{\partial t^2}$ (assuming v_x has negigeble effect on $\frac{\partial h}{\partial t}$, which is one condition on "small perturbation"), corresponding to $\frac{\partial v_z}{\partial t} = \frac{\partial^2 h}{\partial t^2}$ or $-a\omega^2 \exp(kh)\cos(kx - \omega t) = -\omega^2 a \cos(kx - \omega t)$. This is satisfied if $\exp(kh) \approx 1$, which introduces the condition $kh \ll 1$. (For deep-water waves, the other condition $a \ll d$ is trivially satisfied.)

c) [2p] Assume constant pressure p_0 at the surface and show, under the same approximations, the dispersion relation

$$\omega^2 = g_0 k.$$

The definition of Φ^* in (a) gives $g_0 h + \frac{1}{\rho_0} p(x, h, t) = \Phi^*(x, h, t)$. The solution in (b) gives $\Phi^*(x, h, t) = \frac{\omega^2}{k} h \exp(kh) + C$. These are eqal if $\exp(kh) \approx 1$, $C = p(x, h, t)/\rho_0 = p_0/\rho_0$ and $\omega^2/k = g_0$. The last condition gives the dispersion relation.

2. Steady Flows (6p)

Below are three velocity fields for steady, ideal flows, given in non-dimensional variables:

$$\begin{aligned} i) \quad & \boldsymbol{v} = (x,\, -y,\, 0), \\ ii) \quad & \boldsymbol{v} = (-y,\, x,\, 0), \\ iii) \quad & \boldsymbol{v} = (xy,\, yz,\, 0). \end{aligned} \tag{3}$$

a) [3p] Determine which flows that are incompressible, and which flows that are irrotational.

Incompressible flows have $\nabla \cdot \boldsymbol{v} = 0$, which is found for (i) and (ii). Irrotational flows have $\nabla \times \boldsymbol{v} = 0$, which is found for (i).

b) [1p] For one of the flows the Bernoulli field H is a global constant. Determine which one.

The Bernoulli field is constant along stream lines for steady flow, and a global constant for steady, irrotational flow. Field (i) has a constant H.

c) [2p] One of the velocity fields can be represented by a velocity potential Ψ. Determine which one, and find an expression for Ψ.

The flow must be irrotational to allow for the definition of a velocity potnential. That is possible for flow (i). We have $\frac{\partial \Psi}{\partial z} = 0 \Rightarrow \Psi = \Psi(x, y)$, $\frac{\partial \Psi}{\partial x} = x \Rightarrow \Psi = \frac{1}{2}x^2 + f(y)$, $f'(y) = \frac{\partial \Psi}{\partial y} - 0 = -y \Rightarrow f = -\frac{1}{2}y^2 + C$. So, $\Psi = \frac{1}{2}(x^2 - y^2) + C$. Since only "an expression" for Ψ was requested, the undetermined constant C can be omitted.

3. Barotropic, Geostrophic Catastrophe (6p)

For a barotropic atmosphere, we can define a pressure potential $\omega(p)$ obeying $\omega'(p) = \frac{1}{\rho(p)}$.

a) [2p] Use the pressure potential and $\boldsymbol{g} = -\nabla\Phi$, to show that the velocity field for a barotropic atmosphere in geostrophic balance satisfies

$$\nabla \times (2\boldsymbol{\Omega}_z \times \boldsymbol{v}) = 0.$$

The chain rule $\frac{1}{\rho}\nabla p = \omega'(p)\nabla p = \nabla\omega$ and the gravitational potential rewrites the geostrophic balance to $0 = -\nabla\Phi - \nabla\omega - 2\boldsymbol{\Omega}_z \times \boldsymbol{v}$. Taking the curl of this, and using $\nabla \times \nabla = 0$ we get the result.

b) [1p] Assume $\boldsymbol{\Omega}_z$ to be constant and pointing in the vertical direction $\hat{\mathbf{e}}_z$. Look at the separate components of the equation above and show that

$$\frac{\partial v_x}{\partial z} = 0, \quad \frac{\partial v_y}{\partial z} = 0.$$

First x component: $0 = [\nabla \times (\hat{\mathbf{e}}_z \times \boldsymbol{v})]_x = \frac{\partial}{\partial y}[\hat{\mathbf{e}}_z \times \boldsymbol{v}]_z - \frac{\partial}{\partial z}[\hat{\mathbf{e}}_z \times \boldsymbol{v}]_y$. Since a cross product involving $\hat{\mathbf{e}}_z$ is perpendicular to the z axis, the first term vanishes, and we get $0 = -\frac{\partial}{\partial z}[\hat{\mathbf{e}}_z \times \boldsymbol{v}]_y = -\frac{\partial}{\partial z}(v_x - 0)$ so $\frac{\partial}{\partial z}v_x = 0$. Similarly, the y component of the full epxression gives $\frac{\partial}{\partial z}v_y = 0$.

c) [3p] According to (b), the horizontal wind does not change with height! The no-slip condition at ground therefore implies that the horizontal wind must be zero for all z. Thus, winds cannot exist...

Obviously, our initial assumptions have been stretched too far. Discuss what effects near ground that have been neglected in the equation for geostrophic balance, and in qualitative terms how they affect the wind.

> Most important is to consider friction (or viscosity), which creates a boundary layer interpolating between ground and a region of geostrophic balance. The key "qualitative" way it affects the wind is that it creates a component of the wind going (on average, in case of turbulence) in the direction of the pressure gradient force.
>
> The topics above were considered enough for a full score. If some of it was lacking, points could be rewarded for more subtle information, *e.g.*: if the simple assumptions of Ekman holds, the wind as a function of height will "spiral"; to explain data, we must include the contribution of turbulence on the effective viscosity.

4. Sliding on Water (11p)

A box is sliding on a layer of water. The box moves with a velocity $U(t)$. The water has thickness h. The water meets ground at $z = 0$. The ground is at rest.

The box moves in the x-direction, and we assume that the flow in the water is described by

$$\boldsymbol{v}(z,t) = (f(z,t),\, 0,\, 0).$$

a) [1p] Show that this is consistent with a constant water density ρ_0.

> It is, since $\nabla \cdot \boldsymbol{v} = \frac{\partial}{\partial x} f(z,t) + 0 + 0 = 0$.

b) [2p] Assume that water is a Newtonian fluid with constant viscosity, so that the flow obeys the Navier–Stokes equations. With gravity in the z direction and pressure depending only on z, simplify the Navier–Stokes equations as much as possible, and show that f must solve the diffusion equation, $\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial z^2}$. Relate the diffusion constant D to constants in Navier–Stokes equations.

The advective term is $(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = f\frac{\partial}{\partial x}\boldsymbol{v}(z,t) = 0$, and we have $\nabla\cdot v = 0$ from (a). Navier–Stokes reduces to

$$\frac{\partial \boldsymbol{v}}{\partial t} = \boldsymbol{g} - \frac{1}{\rho_0}\nabla p + \nu\nabla^2\boldsymbol{v}.$$

The y component gives trivially $0 = 0$. The z component gives $0 = -g_0 - \frac{1}{\rho_0}\frac{\partial p}{\partial z}$. The x component gives (using the stated properties of $\boldsymbol{g}$ and p) $\frac{\partial f}{\partial t} = \nu\frac{\partial^2 f}{\partial z^2}$. In the last step we have used that $\nabla^2 f(z,t) = \frac{\partial^2 f}{\partial z^2}$. This is the diffusion equation with $D = \nu$.

c) [2p] The box has a contact area S with the water, and will experience a drag force $F = S\sigma_{xz}$. Here, σ_{xz}, taken at $z = h$, is the shear force in the x-direction acting on the box bottom (whose surface is perpendicular to the z-direction). The box has finite mass M, and will be slowed down by the drag force, so that $M\frac{\mathrm{d}U}{\mathrm{d}t} = -F$. Show that f must solve

$$\frac{\partial f(h,t)}{\partial t} = -\alpha\frac{\partial f(h,t)}{\partial z}.$$

and determine the constant α. (Here,$\frac{\partial f(h,t)}{\partial z}$ means $\frac{\partial f}{\partial z}$, taken at $z = h$.)

We have $\sigma_{xz} = \eta\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right) = \eta\frac{\partial f}{\partial z}$. The boundary condition at the box bottom is $f(h,t) = U(t)$, so that $M\frac{\partial f(h,t)}{\partial t} = -S\eta\frac{\partial f(h,t)}{\partial z}$. Divide by M to get $\alpha = S\eta/M$.

d) [3p] Show that

$$f(z,t) = \exp(-s_k t)[A_k\cos(kz) + B_k\sin(kz)],$$

is a possible solution to the diffusion equation, and determine how s_k depends on k. Then, use boundary conditions to determine A_k, and to show that k must satisfy

$$kh\tan(kh) = \varepsilon.$$

Specify the dimensionless constant ε.

Apart from the value of ε, this problem can be solved independently of (c).
We get $\frac{\partial f}{\partial t} = -s_k f$ and $\nu\frac{\partial^2 f}{\partial z^2} = -\nu k^2 f$ which solves the diffusion equation with $s_k = \nu k^2$.
Boundary condition for the floor at rest is $0 = f(0,t) = \exp(-s_k t)A_k$, so that $A_k = 0$. (Side note: since there are in the end many possible k, one might ask if this condition should be relaxed to $\sum_{k_n}\exp(-\nu k_n^2 t)A_{(k_n)} = 0$, but since this should hold for every t, we get $A_k = 0$ for every k.)
Using the result in (c) (even if we failed to show it) gives $-s_k\exp(-s_k t)B_k\sin(kh) = -\alpha k\exp(-s_k t)B_k\cos(kh)$ so that $\tan(kh) = \alpha k/s_k = \alpha/(\nu k)$ and $kh\tan(kh) = \alpha h/\nu$. With α from (c) we get $\varepsilon = S\eta h/M\nu = S\rho_0 h/M$, where ρ_0 is the water density.

e) [1p] We can safely assume that the mass of the water under the box is much smaller than the mass of the box. Use this to motivate that $\varepsilon \ll 1$.

We have ε as the ratio of the water mass and box mass, and so according to the problem it is $\ll 1$.

f) [1p] One solution is then $k_0 \approx \frac{\sqrt{\varepsilon}}{h}$, (since $k_0 h \ll 1$ gives $\tan(k_0 h) \approx k_0 h$) but there are also solutions $k_n \approx n\pi/h$. Use the expression for s_k to motivate why only the solution with k_0 is interesting for large times t.

For this problem, we only need to know s_k from (b). Since the exponential dampening $\exp(-s_k t)$ is stronger for larger s, only the smallest s will matter at large times. With $s \propto k^2$ that implies the smallest k.

g) [1p] To determine the final unknown constants B_k, we must look at initial conditions at $t = 0$. Assume $U(t = 0) = U_0$. Furthermore, assume that the initial condition for the water is such that all $B_{(k_n)}$ are zero, except for k_0. With the approximation $\sin(k_0 z) \approx k_0 z$, determine the constant $B_{(k_0)}$.

The only thing we need to solve this problem is $A_k = 0$, and relations stated in the text. We get $f(z, 0) = B_{(k_0)} \sin(k_0 z) \approx B_{(k_0)} k_0 z$ so that $U_0 = f(h, 0)$ gives $B_{(k_0)} = U_0/kh$.