Solutions to Exam, FYTA14, 2017-06-02

Allowed material: One a4 sheet with notes, writing material.

30 points total, 15 points to pass, 24 points for distinction.

These solutions are in general too brief to give a full score. They are meant to help students reconstruct a good solution.

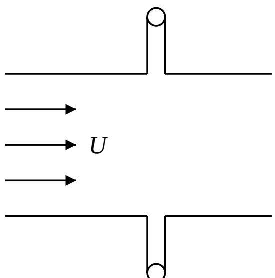
1. Kitchen water velocity (6p)

A friend wants to measure the water velocity in a horizontal pipe, without any disrupting Pitot tube or bottle-neck. My – I mean, *his* – idea is to measure pressure differences due to the Coriolis force.

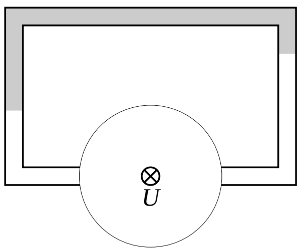
In the following, assume a water velocity $U \sim 1\,\mathrm{m/s}$ and a pipe diameter $\sim 1\,\mathrm{dm}$.

a) [1p] Use order-of-magnitude estimates to discuss if you think Coriolis effects will be measurable in Lund.

With diameter D, the Rossby number for the system is $U/2\Omega D \sim 1/(10^{-5} \cdot 0.1) = 10^6$. The advective term should dominate the Coriolos term.



Picture from above



Picture from inlet

The brilliant(?) idea is to attach a small tube as shown in the figure. The tube is in part filled with water with density ρ_0, and in part (the upper, grey region in the figure) with a fluid with density $\rho_1 < \rho_0$.

b) [3p] A pressure difference between the two sides of the pipe will create a height difference of the surfaces between ρ_0 and ρ_1, in the two vertical parts of the thin tube. Assuming geostrophic balance, estimate how small the *relative density difference*, $(\rho_0 - \rho_1)/\rho_0$ must be for the Coriolis effect to give a $1\,\mathrm{cm}$ height difference.

Geostrophic balance is $0 = \boldsymbol{g} - \frac{1}{\rho_0}\nabla p - 2\boldsymbol{\Omega_z} \times \boldsymbol{v}$. The pressure gradient is perpendicular to $\boldsymbol{v}$ and constant, so the pressure difference across the horizontal diameter of the pipe is $|\Delta p| = \rho_0 2\Omega_z UD$. In the thin tube, there is a height difference h between the two surfaces. Hydrostatic equilibrium then creates a pressure difference $|\Delta p| = (\rho_0 - \rho_1)g_0 h$. Equating them gives $(\rho_0 - \rho_1)/\rho_0 = 2\Omega_z UD/g_0 h \sim 10^{-5}$.

c) [1p] We should however worry what effects we end up measuring. Suppose there is a small velocity difference u between two sides of the big pipe. For lack of better models, assume constant Bernoulli field H. What u would give roughly the same pressure difference as the one from the Coriolis effect? What is your final verdict about my (. . . *friend's*) idea?

At constant height, assuming constant H gives $\Delta p = \rho_0 \frac{1}{2}[(U+u)^2 - U^2] \approx \rho_0 u U$ (anticipating a result $u \ll U$). Equating this with the Coriolis effect gives $\rho_0 u U \approx \rho_0 2\Omega_z U D$ or $u \sim 2\Omega_z D \sim 10^{-6}\,\mathrm{m/s}$. Imperfections in the flow in the pipe would completely overshadow the Coriolis effect.

d) [1p] By the way, what *are* the conditions for constant H?

Most generally: Ideal, incompressible, steady and irrotational. The question could be interpreted as asking for properties not obviously assumed in the rest of problem 1. Thus, answer "steady and irrotational" was accepted as correct. In the general case, the "incompressible" condition can be relaxed if you let H be defined by a pressure potential $\frac{1}{\rho_0} p \to \omega$, where $\omega'(p) = \frac{1}{\rho}$.

2. Flow in a kitchen pipe (6p)

Below are three velocity fields for a flow in the x-direction (along $\widehat{\mathbf{e}}_x$), defined in the domain $y^2 + z^2 \leq R^2$ (a cylindrical pipe of radius R around the x axis):

$$
\begin{aligned}
i) \qquad & \boldsymbol{v} = \exp(-t/\tau) U \widehat{\mathbf{e}}_x, \\
ii) \qquad & \boldsymbol{v} = \left(1 - \frac{y^2 + z^2}{R^2}\right) U \widehat{\mathbf{e}}_x, \\
iii) \qquad & \boldsymbol{v} = \left(1 - \frac{y^2 + z^2}{R^2}\right) \exp(-x/R) U \widehat{\mathbf{e}}_x.
\end{aligned}
$$

Here, $U > 0$ and $\tau > 0$ are constants of apropriate dimensions.

a) [2p] For each field, determine if it is steady, and if it is irrotational.

Field (ii) and (iii) are steady (no time-dependence), and field (i) is irrotational ($\nabla \times \boldsymbol{v} = 0$).

b) [2p] Show that only field (*ii*) can represent a *viscous* flow of an incompressible fluid, inside a pipe at rest with radius R around the x axis.

Only field (*i*) and (*ii*) are incompressible. Only field (*ii*) and (*iii*) satisfy the no-slip boundary condition $\boldsymbol{v}(y^2 + z^2 = R^2) = 0$.

c) [2p] Assuming constant density ρ_0, constant viscosity η, and gravity along $\widehat{\mathbf{e}}_z$, show that the pressure drop $|\Delta p|$ over a distance Δx in the pipe with field (*ii*) is

$$|\Delta p| = 4\eta\Delta x\frac{U}{R^2}.$$

Since $(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = 0$ for field (*ii*), the Navier–Stokes equations boil down to $0 = \boldsymbol{g} - \frac{1}{\rho_0}\nabla p + \nu\nabla^2\boldsymbol{v}$. The x component gives $\frac{\partial p}{\partial x} = \rho_0\nu\nabla^2 v_x = \eta\left(\frac{\partial^2}{\partial y^2}v_x + \frac{\partial^2}{\partial z^2}v_x\right) = -\eta\frac{2+2}{R^2}U$. Integration gives $p = -\eta\frac{4}{R^2}Ux + A(y,z)$. The unknown function A cancels when we take the difference between points in the x-direction, giving the answer. The absolute sign in $|\Delta p|$ was introduced to make sure students did not spend time thinking about directions. It served its purpose. More proper would then be to also change $\Delta x U \to |\Delta x U|$.

3. Tapping kitchen water (6p)

What is the shape of the water jet pouring down from a kitchen tap into a sink?

Assume steady, ideal flow; constant density ρ_0; constant gravitational acceleration $-g_0$ along the vertical direction $\widehat{\mathbf{e}}_z$; and a cylindrical symmetry of the water jet around the vertical. (So, not a tilted jet of water as in the picture...).

Suitable coordinates are height z above the sink bottom, and horizontal distance r from the jet center. Let the horizontal radius of the jet at height z be $a(z)$. As seen in the picture $|a'(z)| \ll 1$, so we assume $|v_r| \ll |v_z|$, where the velocity field is $\boldsymbol{v} = v_z(z,r)\widehat{\mathbf{e}}_z + v_r(z,r)\widehat{\mathbf{e}}_r$.

a) [3p] We may also assume that the pressure is constant in the entire water jet, and that v_z is independent of r. Under all assumptions, show that $a(z) \propto (h - z)^{-1/4}$, where h is an undetermined constant.

Using in succession: Leonardo's law; v_z indep. of r; $|v_r| \ll |v_z|$; H constant along stream lines; constant pressure, we get $\frac{1}{a^4} \propto \langle v_z \rangle^2 = v_z^2 \approx \boldsymbol{v}^2 = C - 2g_0 z - 2p/\rho_0 = \hat{C} - 2g_0 z$.

b) [2p] Assume constant air pressure p_0, and that air does not create any shear forces. Looking at boundary conditions, motivate (wihtout rigorous proofs) the assumptions that pressure is constant in the entire water jet, and that v_z is independent of r.

Neglecting v_r-terms (since $|v_r| \ll |v_z|$) suggests $\frac{\partial p}{\partial r} = 0$. Constant air pressure at the boundary, for all z, then gives constant p. Neglecting $\frac{\partial v_r}{\partial z}$, the lack of shear forces at the boundary suggests $\frac{\partial v_z}{\partial r} = 0$. Since viscosity within the jet would smooth out velocity differences, we have reason to assume $\frac{\partial v_z}{\partial r} = 0$ also in the interior.

c) [1p] The parameter h represents a critical height where there is no energy left for vertical motion. As $v_z \to 0$, the correct total vertical flow can only be obtained if the area $\pi a^2 \to \infty$. Without calculations, motivate how you think corrections due to v_r would modify the critical height.

When $a^2 \to \infty$, we must assume $a' \gg 1$, and v_r matters. When some of the available energy is spent on radial motion, less is available for potential energy ($\propto z$) and vertical motion. Thus, nothing is left for v_z at a lower height, and the critical h is reduced. Note that when p is assumed constant, then constant H implies energy conservation.

One student made the beautiful observation that the same shape would appear with a fountain shooting water vertically upwards, and that h then would represent the height were the water bends over and starts to fall down again.

4. Flow in a kitchen sink (6p)

Look at the bottom of the sink. There is a sudden change from thin, radial flow to much thicker, less orderly flow. Watching the flow in real life shows that the "edge" is stationary. Can we understand it?

At radial distance r from the jet, the water depth $d(r)$ and the average radial velocity, $\langle v_r \rangle(r)$, must obey

$$r\, d\, \langle v_r \rangle = C, \qquad (1)$$

where C is a constant. We do not expect that $d(r)$ or $\langle v_r \rangle(r)$ increase with r. Thus, a reasonable model is that $d(r) \propto r^{-\alpha}$, where $0 \leq \alpha \leq 1$.

a) [2p] In the inner, thin region, consider how much water that in time Δt passes through a vertical area with height $d(r)$ at the circumference of a circle of radius r, and verify eq. (1). (You may of course use eq. (1) below even if you did not verify it.)

The length of the wall at the circumference is $2\pi r$, the height $d(r)$. Water passing through in time Δt is then $2\pi r d(r) \langle v_r \rangle(r) \Delta t$. Since no water accumulates between different radii, this must be independent of r. Which of course is just Leonardo's law on radial flow.

b) [2p] Near the walls of the sink, the flow is perturbed in many ways, creating surface gravity waves. The dispersion relation for a wave with wavelength $\lambda = 2\pi/k$ traveling on water of depth d can be assumed to be $\omega^2(k) = g_0 k \tanh(kd)$ (where g_0 is the gravitational acceleration).

How does the the phase velocity (celerity) c depend on wavelength for shallow-water waves and deep-water waves, respectively?

$c = \omega/k \approx \sqrt{g_0 d}$ for shallow-water waves ($d \ll \lambda \Rightarrow kd \ll 1 \Rightarrow \tanh(kd) \approx kd$) and $\approx \sqrt{g_0/k}$ for deep-water waves ($kd \gg 1 \Rightarrow \tanh(kd) \approx 1$).

c) [2p] The celerity is defined relative to the *water surface*, which itself moves outwards with a velocity that can be assumed to be proportional to $\langle v_r \rangle$. Waves moving inwards will then have a velocity $A\langle v_r \rangle - c$ relative to the *kitchen sink*. (A is a positive constant.) Show that shallow-water waves will stay still relative to the sink at a critical radius R, which depends on unknown constants but is independent of wavelength. Furthermore, show that for $\alpha < 2/3$, waves at $r > R$ move inwards, towards R.

> $A\langle v_r\rangle - c \propto \frac{B}{r^{1-\alpha}} - r^{-\alpha/2} = f(r)[B - r^{1-\frac{3}{2}\alpha}]$, where $f(r)$ is a positivie function of r (since $r > 0$). The critical radius solves $B = R^{1-\frac{3}{2}\alpha}$. Thus $A\langle v_r\rangle - c = g(r)\left[1 - \left(\frac{r}{R}\right)^{1-\frac{3}{2}\alpha}\right]$, where $g > 0$. For $\alpha < 2/3$ and $r > R$, the velocity relative to the sink is negative.

Comment: So, all waves moving inwards from $r > R$ slow down and pile up to form a "sharp edge" at R, which is what we see!

5. Talking about kitchen water... (6p)

Tapping kitchen water creates a lot of funny sounds, propagating in the water. A lecturer once started to talk unprepared about the speed of sound in water, and made a mess of it. Let's help him out!

Look at small, time-dependent perturbations $\Delta\rho$ and Δp to constant values ρ_0 and p_0 of a steady solution. Thus, the fields are $\rho(\boldsymbol{r}, t) = \rho_0 + \Delta\rho(\boldsymbol{r}, t)$ and correspondingly for p. Assuming $|\Delta\rho| \ll \rho_0$ and similarly for p, the linearized differential equations (neglecting gravity) of fluid dynamics become

$$\begin{aligned} \frac{\partial \Delta\rho}{\partial t} + \rho_0 \nabla \cdot \boldsymbol{v} &= 0, \\ \frac{\partial \boldsymbol{v}}{\partial t} &= -\frac{1}{\rho_0}\nabla(\Delta p). \end{aligned}$$

a) [3p] Show that these equations can be combined to a wave equation for density:

$$\frac{\partial^2 \Delta\rho}{\partial t^2} = c_0^2 \nabla^2 \Delta\rho,$$

and describe how c_0 can be determined from a barotropic relation $p = p(\rho)$, where $p_0 = p(\rho_0)$.

> Linearizing $\Delta p \approx p'(\rho_0)\Delta\rho$ gives $\frac{\partial^2\Delta\rho}{\partial t^2} = -\rho_0 \frac{\partial}{\partial t}\nabla \cdot \boldsymbol{v} = -\rho_0 \nabla \cdot \frac{\partial}{\partial t}\boldsymbol{v} = \nabla \cdot (p'(\rho_0)\nabla\Delta\rho) = p'(\rho_0)\nabla^2\Delta\rho$. We get $c_0^2 = p'(\rho_0)$.

b) [1p] The *bulk modulus* $K(\rho)$ of a fluid measures how much pressure increases as a function of relative volume decrease (or relative density increase). It is defined in terms of density and pressure by

$$K(\rho) = \rho \frac{\mathrm{d}p}{\mathrm{d}\rho}.$$

For water, a good parametrization of K is

$$K(\rho) = K_0 + \gamma[p(\rho) - p_0]. \tag{2}$$

In this model, find an expression for c_0. In particular, show that c_0 is independent of the constants p_0 and γ, but depends on the constants K_0 and ρ_0.

> From the definition $K(\rho) = \rho p'(\rho)$ we get $p'(\rho_0) = K(\rho_0)/\rho_0$, and the parametrization of K gives $p'(\rho_0) = K_0/\rho_0$.

c) [1p] If you failed to solve the above sub-problems, use the information that c_0 depends only on K_0 and ρ_0, and apply dimensional arguments to make an educated guess of what c_0 should be. If you *did* solve (a) and (b), make the dimensional arguments anyway, to score your exam point.

> K_0 is a force per area, and ρ_0 is a density. Time dimensions are only found in K_0, and demands $c_0 \propto \sqrt{K_0}$. The only way to get rid of the mass dimension in K_0 is to have $c_0 \propto \sqrt{K_0/\rho_0}$. Checking lengths then shows that this indeed is a velocity. The educated guess is to assume the unknown dimensionless constant to be roughly 1.

d) [1p] Given values $K_0 \sim 2 \cdot 10^9\,\text{Pa}$ and $\rho_0 \sim 10^3\,\text{kg/m}^3$, make a rough estimate of the speed of sound in water.

> Plug in to get $c_0 \sim 1.4 \cdot 10^3\,\text{m/s}$, about 4 times that of sound in air.