Solutions to Exam, FYTA14, 2018-06-01

Allowed material: One a4 sheet with notes, writing material.

30 points total, 15 points to pass, 24 points for distinction.

These solutions are in general too brief to give a full score. They are meant to help students reconstruct a good solution.

1. Non-dimensional (6p)

Below are three three-dimensional velocity fields given in non-dimensional variables:

$$\begin{aligned} &i) \quad \boldsymbol{v} = (tx, -ty, 0), \\ &ii) \quad \boldsymbol{v} = (xy + xz, -yz, 1 - yz), \\ &iii) \quad \boldsymbol{v} = (\cos x, \sin y, t). \end{aligned} \tag{1}$$

a) [1p] For each one of them, determine if it represents steady flow or not.

> Only (*ii*) is steady (lacks explicit t-dependence).

b) [2p] For each one of them, determine if it represents irrotational flow or not.

> Both (*i*) and (*iii*) are irrotational, sovling $\nabla \times \boldsymbol{v} = 0$.

c) [3p] Could any of them be the velocity field for an incompressible ideal flow in the domain $z > 0$, bounded from below by a stationary wall at $z = 0$? If so, compute the corresponding pressure at zero gravity, with the constant non-dimensional density $\rho_0 = 2$.

> Only (*i*) satisfies the boundary condition $v_z(z = 0) = 0$. It also satisfies incompressibility, $\nabla \cdot \boldsymbol{v} = 0$. Since $\partial \boldsymbol{v}/\partial t \neq 0$, the Bernoulliy field is not constant even along stream lines. Instead solving Euler's equation explicitly with $\rho_0 = 2$ gives $\frac{1}{2}\nabla p = (-x - xt^2,\ y - yt^2,\ 0)$ with solution $p = -x^2(1 + t^2) + y^2(1 - t^2) + p_0$, where p_0 is an undetermined constant.

2. Steadily Down the Drain [6p]

Water flows vertically down a rectangular shaft. The shaft is much wider in one direction, so we approximate it as two parallell planes, separated by a distance $2d$ in the x-direction, and disregard the boundary conditions at the walls far away in the y-direction. Looking for a steady solution, we can then assign $\boldsymbol{v} = (0,\, 0,\, v_z(x))$.

Assuming constant pressure, and gravity $\boldsymbol{g} = (0,\, 0,\, -g_0)$, find the velocity $v_z(x)$ that solves the Navier–Stokes equation and satisfies boundary conditions at the considered walls. Let them be at $x = d$ and $x = -d$, respectively.

> The flow $\boldsymbol{v}$ satisfies $\nabla \cdot \boldsymbol{v} = 0$ and $D\boldsymbol{v}/Dt = 0$. With constant pressure, only the z-component of Navier–Stokes equations is interesting, and becomes $0 = -g_0 + \nu v_z''(x)$. This integrates to $v_z(x) = \frac{g_0}{2\nu}x^2 + Ax + B$, and $v_z(\pm d) = 0$ determines constants to $v_z = \frac{g_0}{2\nu}(x^2 - d^2)$.

3. Half Empty (7p)
A large, open tank with constant cross-section A_0 and height h is filled with water. At time $t = 0$, an outlet with cross-section $A < A_0$ is opened at the bottom, and water exits with a velocity v, while the surface in the tank sinks with a velocity v_0.

a) [1p] For steady, ideal flow and constant gravity, find a relation between v, v_0 and z, where z is the height difference between the outlet and the water surface.

> H constant along stream lines and $p = p_0$ wherever there is contact with air means $\frac{1}{2}v^2 = \frac{1}{2}v_0^2 + g_0 z$, where $z = 0$ is chosen at the outlet.

b) [1p] Assuming incompressible flow, find a relation between v and v_0, independent of z.

> Leonardo's law gives $v_0 A_0 = vA$.

c) [1p] Combining the above assumptions, express the velocity through the outlet as a function of z.

> Solving for v gives $v = \sqrt{2g_0 z/(1 - \lambda^2)}$, where $\lambda = A/A_0$.

d) [3p] Allow the water surface height $z(t)$ to vary with time, $\frac{\mathrm{d}z}{\mathrm{d}t} = -v_0$, $z(0) = h$, despite the assumption of steady flow (a "quasi-stationary" approach). Find $z(t)$. What is $T_{1/2}/T$, where $T_{1/2}$ is the time required for the tank to empty halfway and T the time required for the tank to empty completely?

> We get $\frac{\mathrm{d}z}{\mathrm{d}t} = -v_0 = -\lambda v = -C\sqrt{z}$, where C is determined with help of (c). We get $\frac{1}{\sqrt{z}}\mathrm{d}z = -C\mathrm{d}t \Rightarrow 2\sqrt{z} = -Ct + A$, where $A = 2\sqrt{h}$ solves the initial condition $z(t = 0) = h$. We get $T_{1/2}/T = t(z = h/2)/t(z = 0) = (\sqrt{h} - \sqrt{h/2})/\sqrt{h} = 1 - 1/\sqrt{2}$.

e) [1p] Hopefully, your results in (c) and (d) are un-physical in the limit $A \to A_0$, representing a situation where almost the whole bottom of the tank is opened as an outlet. What assumption(s) above would be un-reasonable for large A? (It is possible to answer this question without any answers in (c) and (d) to refer to.)

> For $\lambda \to 1$, v diverges. In a free fall scenario the steady flow (or quasi-stationary flow) is unreasonable. Mentioning other approximations as unreasonable can also score points, if well motivated.

4. Tilted (6p)
Assume a steady, uniform current of water flowing horizontally with speed U, so that the velocity field is $\boldsymbol{v} = U\widehat{\boldsymbol{e}}_x$, where the x-direction is defined along the current.

Due to the Coriolis force, the water surface will not be quite horizontal, despite a vertical gravitational field $\boldsymbol{g} = -g_0\widehat{\boldsymbol{e}}_z$.

a) [1p] Where will the slope be most noticeable (assuming no disturbances such as winds, or boats, or waves, or...), in Öresund between Sweden and Denmark, or in the Suez canal in Egypt? Motivate your answer.

> The slope results from $\boldsymbol{\Omega}_z \times \boldsymbol{v}$, and Ω_z is small near the equator. The effect is largest in Öresund. One could argue that the width of the flow matters, since the slope most likely will be measured as a height difference between the banks. If so, the Öresund slope is also more noticeable.

b) [4p] Assuming geostrophic balance, incompressible water and constant air pressure, find an expression for the water surface. You will need to introduce additional symbols for various physical quantitites.

The proposed $\boldsymbol{v}$ and $\boldsymbol{g}$ simplify the equation for geostrophic balance to $\frac{\partial p}{\partial z} = -\rho_0 g_0$, $\frac{\partial p}{\partial y} = -2\rho_0 \Omega_z U$, with solution $p = C - \rho_0 g_0 z - 2\rho_0 \Omega_z U y$. The surface is described by a function $z = h(y)$ and satisfies the pressure boundary condition $p(y, h) = p_0$, where air pressure is assumed constant. This gives $h(y) = \frac{C - p_0}{\rho_0 g_0} - \frac{2\Omega_z U}{g_0} y = h_0 - \frac{2\Omega_z U}{g_0} y$, where the unknown C has been replaced by a more interpretable constant h_0.

c) [1p] A reasonable water velocity is $U \sim 1\,\text{m/s}$, and a uniform flow can hardly be a good approximation at length scales above $L \sim 10^4\,\text{m}$. Near Lund, the local angular velocity is $\Omega_z \sim 0.6 \cdot 10^{-4}\,\text{s}^{-1}$. Estimate the Rossby number for the system. Does it motivate consideration of the Coriolis force? Is the Rossby number relevant for the proposed flow?

The Rossby number is $UL/2\Omega_z \sim 1$, but the proposed flow $\boldsymbol{v} = U\widehat{\boldsymbol{e}}_x$ has $(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = 0$, so U and L cannot be used to estimate the importance of the advective term.

5. Shallow (5p)
Consider ideal water flow in constant gravity $\boldsymbol{g} = -g_0 \widehat{\boldsymbol{e}}_z$. Assume constant water density ρ_0.

a) [1p] Show that if the advective term can be neglected, Euler's equation becomes

$$\frac{\partial}{\partial t}\boldsymbol{v}(\boldsymbol{r}, t) = -\nabla\left[\Phi(z) + \frac{1}{\rho_0} p(\boldsymbol{r}, t)\right],$$

where p is the pressure. Give an expression for the function $\Phi(z)$.

Neglecting $(\boldsymbol{v} \cdot \nabla)\boldsymbol{v}$ and finding $\Phi(z) = g_0 z$ gives the expression.

b) [2p] Show that $\Phi^* = \Phi + p/\rho_0$ satisfies the Laplace equation $\nabla^2 \Phi^* = 0$.

Incompressibility $\nabla \cdot \boldsymbol{v} = 0$ gives $\nabla^2 \Phi^* = -\nabla \cdot \frac{\partial \boldsymbol{v}}{\partial t} = -\frac{\partial}{\partial t}\nabla \cdot \boldsymbol{v} = 0$.

The water surface height h is described by a harmonic wave of amplitude a moving in the x direction, $h = a\cos(kx - \omega t)$, where $k > 0$ and $\omega > 0$ are constants. We therefore expect

$$\Phi^*(x, z, t) = f(z)\cos(kx - \omega t + \theta) + C,$$

where θ and C are constants and $f(z)$ is an unknown function of z.

c) [2p] Find θ and $f(z)$ so that Φ^* solves the Laplace equation and the derivative $f'(z)$ satisfies the boundary conditions $f'(-d) = 0$ and $\frac{\partial}{\partial z}\Phi^*(x,z,t)\big|_{z=0} = kA\sinh(kd)\cos(kx-\omega t)$, where A is an undetermined constant. Consider the range $-d \leq z \leq 0$ and show that the shallow water limit, $kd \ll 1$, makes $f'(z)$ approximately a linear function of z. (Note: you do not need to motivate the given boundary conditions. That's for another exam.)

We get $0 = \nabla^2\Phi^* = (f - k^2 f'')\cos\phi$, where $\phi = kx - \omega t + \theta$. The general solution to f is a linear combination of $\exp(kz)$ and $\exp(-kz)$. Cleverly writing it as $f(z) = C\cosh k(z-a)$ with unknown constants C and a, allows boundary conditions to rapidly infer $a = -d$, $C = A$ and $\theta = 0$. So, $f(z) = A\cosh k(z+d)$ and $f'(z) = kA\sinh k(z+d)$. The range $-d \leq z \leq 0$ and limit $kd \ll 1$ means $|k(z+d)| \ll 1$ so that $\sinh k(z+d) \approx k(z+d)$. That makes $f'(z)$ approximately linear in z.