**Solutions to exam, FYTA14, 2019-06-07**

**Allowed material**: One a4 sheet with notes, writing material.

**30 points total, 15 points to pass, 24 points for distinction.**

The solutions are too brief for full score, to highlight the main arguments. Examples of required sketches are not shown.

## 1. Pressure in a Tornado (9p)

The "Rankine vortex" can be used as a simple model for tornadoes. It assumes the flow to be

$$\boldsymbol{v} = \begin{cases} \boldsymbol{v}^{(core)} & = \quad \omega_0(-y,\, x,\, 0),\ r \leq R \\ \boldsymbol{v}^{(out)} & = \quad \omega_0 \frac{R^2}{r^2}(-y,\, x,\, 0).\ r > R \end{cases} . \tag{1}$$

Here, $\omega_0$ and $R$ are constants, and $r$ is the horizontal distance to the tornado center:

$$r = \sqrt{x^2 + y^2}.$$

Thus, it assumes the velocity to be independent of height $z$.

We will assume that $\boldsymbol{v}$ describes ideal flow in a vertical gravitational field $\boldsymbol{g} = (0,\, 0,\, g_z)$. We also assume constant density $\rho_0$, for simplicity.

***a***) [1p] Show that the z-component of Euler's equation reduces to an equation for hydrostatic equilibrium.

*Solution:* With $v_z = 0$ the z-component of $\frac{\partial}{\partial t}\boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \boldsymbol{g} - \frac{1}{\rho_0}\nabla p$ becomes $0 = g_z - \frac{1}{\rho_0}\frac{\partial p}{\partial z}$ which is the z-component of the equation for hydrostatic equilibrium, with $bsv = 0$.

***b***) [3p] For both the core region ($r \leq R$) and the outer region ($r > R$), determine if the flows are steady, incompressible and/or irrotational, respectively. Be careful to handle product rules and chain rules correctly with the factor $\frac{1}{r^2} = \frac{1}{x^2+y^2}$.

*Solution:* Both are steady (no $t$-dependence). The core region flow is Incompressle $\nabla \cdot \boldsymbol{v} \propto \partial_x(-y) + \partial_y x = 0$, but not irrotational $[\nabla \times \boldsymbol{v}]_z = \partial_x v_y - \partial_y v_x \propto \partial_x x + \partial_y y = 2 \neq 0$. To investigate the outer region without converting to cylindrical coordinates, we note that $\partial_x(x/r^2) = 1/r^2 - 2x^2/r^4$, $\partial_y(x/r^2) = -2xy/r^4$, $\partial_x(y/r^2) = -2xy/r^4$ and $\partial_y(y/r^2) = 1/r^2 - 2y^2/r^4$. This gives $\nabla \cdot (-y/r^2,\, x/r^2) = 2xy/r^4 - 2xy/r^4 = 0$ and $[\nabla \times \boldsymbol{v}]_z = \partial_x v_y - \partial_y v_x = 1/r^2 - 2x^2/r^4 + 1/r^2 - 2y^2/r^4 = 2/r^2 - 2(x^2+y^2)/r^4 = 2/r^2 - 2r^2/r^4 = 0$, so the outer flow is both incompressible and irrotational.

***c***) [1p] Your results in (b) should allow you to conclude that the Bernoulli field $H$ is constant for one of the regions, but not the other. Determine which one. Alternatively, if your results in (b) do not lead to the correct conclusion, discuss what would need to change.

*Solution:* The Bernoulli field is constant for steady, ideal, incompressible, irrotational flow. That is fulfilled in the outer region, but not the core.

***d***) [3p] Calculate the pressure in each of the regions. The answers should contain one undetermined constant each.

*Solution:* For the core we solve Euler's equations explicitly. From $(\boldsymbol{v} \cdot \nabla) = \omega_0^2(-y\partial x + x\partial_y)(-y,\, x) = \omega_0^2(-x,\, -y) = -\omega_0^2 \boldsymbol{r}$ we get $\nabla p = \rho_0 \boldsymbol{g} + \rho_0\omega_0^2 \boldsymbol{r}$. Integrating $\partial_x p$ over $x$ gives $p = \frac{1}{2}\rho_0\omega_0^2 x^2 + A(y,z)$ and $\partial_y A = \partial_y p = \rho_0\omega_0^2 y$ gives $p = \frac{1}{2}\rho_0\omega_0^2 r^2 + B(z)$. Including the vertical $\partial_z p = \rho_0 g_z$ sets $B(z) = \rho_0 g_z z + C$, where $C$ is a constant.

For the outer region, we exploit constant $H$ to get $p = D + \rho_0 g_z z - \rho_0 \frac{1}{2}\boldsymbol{v}^2 = D + \rho_0 g_z z - \frac{1}{2}\rho_0\omega_0^2 R^4/r^2$, where $D$ is a constant.

***e***) [1p] Assuming continuous pressure at $r = R$, show the difference $p(r=0) - p(r=\infty) = -\rho_0\omega_0^2 R^2$.
*Comment:* This illustrates the possibility of very low pressure in the centre of a tornado!

*Solution:* At constant height $z$ the equality at $r = R$ implies $\frac{1}{2}\rho_0\omega_0^2 R^2 + C = D - \frac{1}{2}\rho_0\omega R^4/R^2$ or $C = D - \rho_0\omega_0^2 R^2$. Then $p(0) - p(\infty) = C - D = -\rho_0\omega_0^2 R^2$.

## 2. The Gulf Stream (6p)

The Gulf Stream runs counter-clockwise in the northern Atlantic Ocean (as a very simplified description). According to Wikipedia it is typically 100km wide and flows with a velocity of about 1 m/s. We assume that the current can be described by *geostrophic balance*, and that the water surface lies at constant pressure $p_0$.

***a***) [2p] Describe why there will be a height difference between the inner and outer edges of the current, and determine which edge that has the highest surface. Furthermore, assuming (boldly) that the width and speed of the flow is constant, where on earth would the height difference be maximized?

*Solution:* The Coriolis force in geostrophic balance will act to the right for streams in the northern hemisphere, which follows from the term $-2\Omega_z\widehat{\boldsymbol{e}}_z \times \boldsymbol{v}$. The pressure at the water surface is expected to be fairly constant, and a height difference will then create a pressure gradient which can componesate the Coriolis force. To do so, the surface must be higher in the right edge of the stream. For counter-clockwise motion, that is the outer edge. (This correponds to winds going counter-clockwise around a low pressure. The lower surface on the inner side creates a "low pressure" for water beneath the surface. The current goes counter-clockwise around that low pressure.)

***b***) [2p] Deeper down, the Gulf stream meets other water layers with different motion. At the lower regions of the Gulf stream there is therefore an Ekman layer where the flow disagrees with geostrophic balance. Qualitatively, what is the direction of the flow in the Ekman layer, and how does that affect the height difference between the outer and inner water surfaces?

*Solution:* The simplest argument is that viscosity will change the velocity to become smaller, and then the force from the pressure gradient will dominate. So there will be a velocity component in the direction of negative pressure gradient, inwards. This will reduce the height difference between the outer and inner edge.

Note that the question refers to an Ekman layer at the bottom of the stream, not a layer at the surface, where winds may start a water motion whose direction changes with depth due to Coriolis effects. However, at the surface, the flow is not influenced by any pressure gradient due to water height differences, so by-heart knowledge from one kind of layer does not apply to the other kind.

***c***) [1p] If the stream instead had run clock-wise, with preserved numerical values, would the height difference change in size or direction? Would the effects of the Ekman layer change? For the "yes" answers: what is the change?

*Solution:* The Coriolis effect still drives the flow to the right, so now the inner edge must be the higher (a clock-wise flow around a "high pressure" in the middle), with the same difference magnitude as before. The Ekman layer at the bottom will be influence by the new pressure gradient and go outwards. It will still reduce the height difference.

***d***) [1p] *A side-remark stated as a sub-problem*: When the counter-clockwise Gulf stream turns left, you may expect a pressure gradient to drive the change of velocity direction. With a typical speed $U$ and radius $R$ of the turn, the acceleration "left-wards" is $U^2/R$. With reasonable assumptions on $R$, show that this acceleration is negligible compared to other effects in geostrophic balance. Thus, this acceleration need not be considered in the rest of the problem!

*Solution:* We want to compare the "left-ward" (centripetal) acceleration the Coriolis acceleration effect, so we look at the Rossby number $U^2/R2\Omega U = U/2\Omega R$. A typical $U$ was given as $1\,\mathrm{m/s}$ and the typical radius can hardly be smaller than the width of the stream, so a reasonable upper bound on the Rossby number $\sim 1/(10^{-4}10^5) = 0.1$. A student cleverly pointed out that these 10% cooperate with the Coriolis effect for counter-clockwise flow, but counter-acts it for clockwise flow, so the magnitude of the height difference in the two scenarios is not quite the same. A beautiful remark, which definitely is not needed for full score.

## 3. Flow on a Plane (7p)

Consider steady water flow in the x-direction on a plane perpendicular to the z-direction, such that $\mathbf{v} = (v_x(z),\, 0,\, 0)$. The water surface is at constant height above the plane. Assume constant water density $\rho_0$ and a constant pressure $p_0$ at the surface.

Let the plane (and the coordinate system) be tilted with respect to the earth vertical, so that the gravitational acceleration is $\mathbf{g} = (g_x,\, 0,\, g_z)$, where $g_x$ and $g_z$ are constants.

***a***) [1p] Verify that the proposed velocity field is consistent with a constant density.

*Solution:* Constant density requires $\nabla \cdot \boldsymbol{v} = 0$. Indeed, we have $\nabla \cdot \boldsymbol{v} = \frac{\partial v_x}{\partial x} = 0$, since

$v_x = v_x(z)$.

***b***) [2p] Show that the pressure in the fluid is independent of $x$. *Hint:* Use the z-component of the Navier–Stokes equations and the pressure boundary condition at the surface.

*Solution:* With steady flow and $\nabla \cdot \boldsymbol{v} = 0$, the z-component reads $0 = g_z - \frac{1}{\rho_0}\frac{\partial p}{\partial z}$. Integrating gives $p = \rho_0 g_z z + A(x, y)$, where $A$ is an unknown function of $x$ and $y$. However, the boundary condition at height $z = h$ (introducing $h$), is $p = p_0$, so that $A = p_0 - g_z \rho_0 h$ and independent of $x$ (and $y$).

***c***) [1p] Knowing that the pressure is independent of $x$ (even if you have not proven it), find an expression for $v_x''(z)$.

*Solution:* We look at the x-component of Navier–Stokes, where most terms become zero, so that it remains $0 = g_x + \nu \frac{\partial^2 v_x}{\partial z^2}$. Since $v_x$ only depends on $z$, the partial derivative symbol is superfluous and we find $v_x''(z) = -g_x/\nu$.

***d***) [3p] Find an expression for $v_x(z)$ and determine all integration constants with the help of two boundary conditions:
1) The velocity at the plane
2) The shear forces at the surface. Assume that air imposes negligible shear forces on the water, so that the stress tensor component $\sigma_{xz}$ is 0 at the water surface.

*Solution:* Integrating twice gives $v_x = -\frac{g_x}{2\nu} z^2 + C_1 z + C_2$, where $C_1$ and $C_2$ are constants. Defining coordinate system so that $z = 0$ at the plane gives $v_x = -\frac{g_x}{2\nu} z^2 + C_1 z$ from the no-slip boundary condition (making the reasonable assumption that the plane is still). The stress tensor component $\sigma_{xy}$ is $\eta(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}) = \eta v_x'(z) = \eta(-\frac{g_x}{\nu} z + C_1)$. For this to vanish at the surface we must have $C_1 = \frac{g_x}{\nu} h$ so that $v_x = \frac{g_x}{\nu}\left(zh - \frac{1}{2} z^2\right)$. Since we have defined $z = h$ at the surface and $z = 0$ at the plane, the symbol $h$ represents the height (perpendicular to the tilted plane) of the flow.

Note: An equally reasonable coordinate system is to have $z = 0$ at the surface and $z = -d$ at the plane. Then the final result is $v_x = \frac{g_x}{\nu}(d^2 - z^2)$.

### 4. Sound Approximations (8p)

Consider an ideal, compressible gas with small, time-dependent corrections to a steady solution, so that $\rho = \rho_0(\boldsymbol{r}) + \varepsilon \rho_1(\boldsymbol{r}, t)$, $p = p_0(\boldsymbol{r}) + \varepsilon p_1(\boldsymbol{r}, t)$ and $\boldsymbol{v} = \boldsymbol{v}_0(\boldsymbol{r}) + \varepsilon \boldsymbol{v}_1(\boldsymbol{r}, t)$. Here, a dimensionless number $\varepsilon \ll 1$ has been introduced to emphasize that the time-dependent corrections are small, *e.g.*, $|\varepsilon \rho_1| \ll \rho_0$. To simplify, we assume $\boldsymbol{v}_0 = 0$.

During the course, we encountered an even simpler example, with zero gravity and constant $p_0$, $\rho_0$. Here, we will allow for space dependence $p_0(\boldsymbol{r})$ and $\rho_0(\boldsymbol{r})$ and introduce a constant gravity field $\boldsymbol{g}$.

We assume that there exists a barotropic relation $p = f(\rho)$, and define the static fields so that $p_0(\boldsymbol{r}) = f(\rho_0(\boldsymbol{r}))$.

***a***) [1p] Use Taylor expansion of $f(\rho)$ to show that

$$\rho_1 = \frac{1}{c_0^2} p_1$$

and define the velocity $c_0$.

*Solution:* Comparing $p = p_0 + \varepsilon p_1$ and $p = f(\rho_0 + \varepsilon\rho_1) \approx f(\rho_0) + f'(\rho_0)\varepsilon\rho_1$ gives $p_1 = f'(\rho_0)\rho_1$, so $c_0^2 = f'(\rho_0) = \frac{\mathrm{d}p}{\mathrm{d}\rho}\Big|_{\rho=\rho_0}$.

***b***) [3p] Taylor expand the equation of continuity and Euler's equations in $\varepsilon$, and show that the $\mathcal{O}(1)$ terms become one trivial $0 = 0$ and

$$0 = \boldsymbol{g} - \frac{1}{\rho_0}\nabla p_0, \tag{2}$$

while the $\mathcal{O}(\varepsilon)$ terms result in the equations

$$\begin{aligned} \frac{\partial}{\partial t}\rho_1 + \nabla\cdot(\rho_0 \boldsymbol{v}_1) &= 0, & (3)\\ \frac{\partial}{\partial t}\boldsymbol{v}_1 &= \frac{\rho_1}{\rho_0^2}\nabla p_0 - \frac{1}{\rho_0}\nabla p_1. & (4) \end{aligned}$$

*Solution:* $0 = \frac{\partial}{\partial t}(\rho_0 + \varepsilon\rho_1) + \nabla\cdot[(\rho_0+\varepsilon\rho_1)\varepsilon\boldsymbol{v}_1] = \varepsilon\frac{\partial}{\partial t}\rho_1 + \varepsilon\nabla\cdot(\rho_0\boldsymbol{v}_1) + \mathcal{O}(\varepsilon^2)$ gives $0=0$ and eq. (3). The other equations come from matching orders in $\frac{\partial}{\partial t}\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = \varepsilon\frac{\partial}{\partial t}\boldsymbol{v}_1 + \mathcal{O}(\varepsilon^2)$ with orders in $\boldsymbol{g} - \frac{1}{\rho_0+\varepsilon\rho_1}\nabla(p_0+\varepsilon p_1) = \boldsymbol{g} - \frac{1}{\rho_0}\left(1 - \varepsilon\frac{\rho_1}{\rho_0}\right)\nabla(p_0 + \varepsilon p_1) + \mathcal{O}(\varepsilon^2) = \boldsymbol{g} - \frac{1}{\rho_0}\nabla p_0 + \varepsilon\frac{\rho_1}{\rho_0^2}\nabla p_0 - \varepsilon\frac{1}{\rho_0}\nabla p_1 + \mathcal{O}(\varepsilon^2)$.

***c***) [3p] Study $\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}p_1 = \frac{\partial^2}{\partial t^2}\rho_1$ and show that the three equations stated in (b) combine to

$$\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}p_1 = \nabla^2 p_1 - (\boldsymbol{g}\cdot\nabla)\frac{p_1}{c_0^2}. \tag{5}$$

*Solution:* $\frac{\partial^2}{\partial t^2}\rho_1 = -\frac{\partial}{\partial t}\nabla\cdot(\rho_0\boldsymbol{v}_1) = -\nabla\cdot(\rho_0\frac{\partial}{\partial t}\boldsymbol{v}_1) = -\nabla\cdot\left[\rho_0\left(\frac{\rho_1}{\rho_0^2}\nabla p_0 - \frac{1}{\rho_0}\nabla p_1\right)\right] = -\nabla\cdot\left[\frac{\rho_1}{\rho_0}\nabla p_0 - \nabla p_1\right] = -\nabla\cdot[\rho_1\boldsymbol{g} - \nabla p_1]$. Since $\boldsymbol{g}$ is constant this becomes the equation above.

***d***) [1p] Without the $\boldsymbol{g}$-dependent term, the result in (c) is just a wave equation, and we expect solutions of the form $p_1 \propto \sin(kx - \omega t)$ where $\omega = c_0 k$. Thus, the relative correction due to $\boldsymbol{g}$ can be estimated by $\frac{|\boldsymbol{g}\cdot\nabla p_1|}{|\frac{\partial^2}{\partial t^2}p_1|} \sim \frac{gk}{\omega^2}$. The lower audible angular frequencies $\omega$ for a human ear are about $100\,\mathrm{s}^{-1}$. Use your best estimates (they do not have to be good) about $g$ and $c_0$ to discuss if you think the gravity-dependent term needs to be considered in everyday applications about audible sound.

*Solution:* With $g \sim 10\,\mathrm{m/s^2}$, $\omega > 100\,\mathrm{s}^{-1}$ and $c_0 \sim 300\,\mathrm{m/s}$, we get $\frac{gk}{\omega^2} \sim \frac{g}{c_0\omega} <\sim 10^{-3} \ll 1$ The relative magnitude of the gravity-dependent term makes it negligible.