Solutions to exam, FYTA14, 2020-06-04

Allowed material: One a4 sheet with notes, writing material.

30 points total, 15 points to pass, 24 points for distinction.

The solutions are too brief for full score, to highlight the main arguments.

1. Velocity Fields (4p)

Below are three velocity fields, given in non-dimensional variables:

$$\begin{aligned} i) &\qquad \boldsymbol{v} = (xt,\ -yt,\ t), \\ ii) &\qquad \boldsymbol{v} = (\sin y,\ 3\cos x,\ 0), \\ iii) &\qquad \boldsymbol{v} = (2z - z^2,\ 0,\ 0). \end{aligned} \tag{1}$$

a) [1p] For each flow, determine if it is incompressible.

$\nabla \cdot \boldsymbol{v} = 0$ holds for (i) and (iii). They are incompressible.

b) [1p] For each flow, determine if it is irrotational.

$\nabla \times \boldsymbol{v} = 0$ holds for (i). It is irrotational.

c) [1p] Determine which fields (if any) that are consistent with ideal flow above a floor at $z = 0$.

Flow (ii) and (iii). They satisfy the boundary condition $v_z = 0$ at $z = 0$.

d) [1p] Determine which fields (if any) that are consistent with a steady, viscous flow above a floor at rest at $z = 0$.

Flow (iii). It satisfies the no-slip boundary condition $\boldsymbol{v} = 0$ at $z = 0$, and has no explicit time-dependence.

2. Origin of Geostrophic Flow (5p)

Consider ideal flow expressed in a coordinate frame rotating with constant angular velocity $\boldsymbol{\Omega}$ with respect to an inertial frame.

a) [1p] What terms are then added to Euler's equations for ideal flow?

$-2\boldsymbol{\Omega} \times \boldsymbol{v} - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r})$, where $\boldsymbol{r}$ are the coordinate vector in the rotating frame, and $\boldsymbol{v}$ is the velocity field.

b) [3p] In many earth applications, the $\boldsymbol{\Omega}$-dependent terms can be neglected or simplified. Describe how and give motivations. In particular, discuss which approximations that depend on a typical velocity U and length-scale L of the studied system, and which approximations that are more general.

The centrifugal term $-\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r})$ can be neglected under most earth applications. It is negligible compared to the gravitational acceleration $\boldsymbol{g}$. For the Coriolis term, the angular velocity can be approximated by its vertical part, $\boldsymbol{\Omega}_z$. The two contributions from the horizontal part are $-2\Omega_y v_x$ in the vertical direction, which in most situations is negligible compared to $\boldsymbol{g}$, and $-2\Omega_y v_z$ in a horizontal direction, which most often can be neglected due to the vertical flow v_z being small in most wind and ocean applications.

Taking the typical velocity and length scales into account, we can look at the Rossby number $U/2\Omega_z L$, which indicates the relation between the advective tern $(\boldsymbol{v} \cdot \nabla)\boldsymbol{v}$ and the Coriolis term $-2\boldsymbol{\Omega}_z \times \boldsymbol{v}$. If the Rossby number is large, the Coriolis term can be neglected.

c) [1p] What is the equation for geostrophic flow?

We add the Coriolis term and neglect the advective term: $\frac{\partial}{\partial t}\boldsymbol{v} = \boldsymbol{g} - \frac{1}{\rho}\nabla p - 2\boldsymbol{\Omega}_z \times \boldsymbol{v}$, where ρ is density and p is pressure.

3. Pitot Tube (5p)

A thin, "L-shaped" tube with open ends is lowered into a canal. The opening under water is turned towards the flow in the canal. The water level inside the tube is then observed to be higher than the canal surface. Starting with ideal flow and other reasonable assumptions, derive an expression that relates the height difference h to flow velocity U in the canal.

A figure is needed, but not included here. Using the Bernoulli theorem, we find the pressure difference between the stagnation point S hitting the tube and a point upstream on the same horizontal streamline, where flow is assumed uniform with velocity U, to be $p_S - p_U = \frac{1}{2}\rho_0 U^2$. In the upsteream region, $v_z = 0$ and inside the tube there is hydrostatic equilibrium. Assuming constant air pressure at all water surfaces, the same pressure difference can then be expressed as $p_S - p_U = \rho_0 g_0 (d + h) - \rho_0 g_0 d = \rho_0 g_0 h$, where d is the depth of the tube opening with the stagnation point, and h is the extra water height observed inside the tube. Combining gives $2g_0 h = U^2$.

4. Dampened Sound (7p)

Consider an ideal, compressible gas close to an equilibrium where the velocity field is $\boldsymbol{v}_0 = 0$ and density ρ_0 and pressure p_0 are constants. Let $\boldsymbol{v}$, $\Delta\rho$ and Δp denote the deviations from the equilibrium values, so that $\rho = \rho_0 + \Delta\rho$, etc. Then, the linearized Euler equations become

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\frac{1}{\rho_0}\nabla(\Delta p) \tag{2}$$

and the linearized equation of continuity becomes

$$\frac{\partial \Delta\rho}{\partial t} = -\rho_0 \nabla \cdot \boldsymbol{v}. \tag{3}$$

a) [3p] Suppose the flow obeys a barotropic relation $p = p(\rho)$. Show that $\Delta\rho$ satisfies the wave equation

$$\frac{\partial^2}{\partial t^2}(\Delta\rho) = c_0^2 \nabla^2(\Delta\rho), \tag{4}$$

and describe how c_0 can be determined from the barotropic relation.

$\frac{\partial^2}{\partial t^2}(\Delta\rho) = -\rho_0 \frac{\partial}{\partial t}\nabla \cdot \boldsymbol{v} = -\rho_0 \nabla \cdot (\frac{\partial \boldsymbol{v}}{\partial t}) = -\rho_0 \nabla \cdot (-\frac{1}{\rho_0}\nabla p) = \nabla^2(\Delta p)$. Taylorexpansion of the barotropic relation gives $\Delta p = p - p_0 = p(\rho + \Delta\rho) - p(\rho) \approx (\Delta\rho)p'(\rho_0)$. We define the constant $c_0^2 = p'(\rho_0)$ and move it in front of the Laplace operator to get the wave equation.

b) [3p] Instead consider a viscous fluid, with constant viscosity η and bulk viscosity ζ. The vicuous terms in the the Navier–Stokes equations are linear in $\boldsymbol{v}$, and can immediately be added to the linearized Euler equations eq. (2). Show that this modifies the wave equation to

$$\frac{\partial^2}{\partial t^2}(\Delta\rho) = c_0^2 \left[\nabla^2(\Delta\rho) + \frac{1}{\omega_0}\nabla^2 \frac{\partial}{\partial t}(\Delta\rho)\right], \tag{5}$$

and find an expression for the introduced constant ω_0.

Adding the lienar viscous terms means $\frac{\partial}{\partial t}\boldsymbol{v} = -\frac{1}{\rho_0}\nabla(\Delta p) + \frac{\eta}{\rho_0}\nabla^2\boldsymbol{v} + \frac{\zeta+\eta/3}{\rho_0}\nabla(\nabla\cdot\boldsymbol{v})$ so that $\frac{\partial^2}{\partial t^2}(\Delta\rho) = -\rho_0\nabla\cdot\frac{\partial\boldsymbol{v}}{\partial t} = \nabla^2(\Delta p) - \eta\nabla\cdot(\nabla^2\boldsymbol{v}) - (\zeta+\eta/3)\nabla\cdot[\nabla(\nabla\cdot\boldsymbol{v})]$. Order of differentiation can change as long as we do not change any scalar products, so we have $\nabla\cdot(\nabla^2\boldsymbol{v}) = \nabla^2(\nabla\cdot\boldsymbol{v})$. This gives $\frac{\partial^2}{\partial t^2}(\Delta\rho) = -\rho_0\nabla\cdot\frac{\partial\boldsymbol{v}}{\partial y} = \nabla^2(\Delta p) - \eta\nabla^2(\nabla\cdot\boldsymbol{v}) - (\zeta+\eta/3)\nabla^2(\nabla\cdot\boldsymbol{v}) = c_0^2\nabla^2(\Delta\rho) - (\zeta+4\eta/3)\nabla^2(\nabla\cdot\boldsymbol{v}) = c_0^2\nabla^2(\Delta\rho) + (\zeta+4\eta/3)\nabla^2\frac{1}{\rho_0}\frac{\partial}{\partial t}(\Delta\rho)$. With $\omega_0 = \frac{c_0^2\rho_0}{\zeta+4\eta/3}$ we get the desired equation.

c) [1p] A possible solution to eq. (5) is a dampened sound wave,

$$\Delta\rho = \rho_1\exp(-\kappa x)\cos(kx-\omega t), \tag{6}$$

where ρ_1, κ, k and ω are constants. In the limit of $\omega \ll \omega_0$, determine how the dampening coefficient κ depends on sound frequency ω.

We have $\frac{\partial^2}{\partial t^2}(\Delta\rho) = -\omega^2(\Delta\rho)$ and find after some work $\nabla^2(\Delta\rho) = (\kappa^2-k^2)(\Delta\rho) + 2\kappa k\rho_1\exp(-\kappa x)\sin(kx-\omega t) = (\kappa^2-k^2)(\Delta\rho)+2\kappa kf$, where we for compactness introduce $f(x,t) = \rho_1\exp(-\kappa x)\sin(kx-\omega t)$. We have $\frac{\partial}{\partial t}(\Delta\rho) = \omega f$ and $\frac{\partial}{\partial t}f = -\omega(\Delta\rho)$. Therefore, $\frac{1}{\omega_0}\nabla^2\frac{\partial}{\partial t}(\Delta\rho) = \frac{1}{\omega_0}\frac{\partial}{\partial t}[(\kappa^2-k^2)(\Delta\rho)+2\kappa kf] = \frac{\omega}{\omega_0}[(\kappa^2-k^2)f - 2\kappa k(\Delta\rho)]$. Putting all together gives $-\frac{\omega^2}{c_0^2}(\Delta\rho) = (\kappa^2-k^2)(\Delta\rho) - \frac{\omega}{\omega_0}2\kappa k(\Delta\rho) + 2\kappa kf + \frac{\omega}{\omega_0}(\kappa^2-k^2)f$. The f-dependent terms must cancel, since they behave as $\sin(kx-\omega t)$ rather than the $\cos(kx-\omega t)$ behaviour on the right-hand-side of the equation. Therefore, $2\kappa k = \frac{\omega}{\omega_0}(k^2-\kappa^2)$. For a wave damped in the direction of its own propagation, we have $\kappa > 0$ and $k > 0$. To get the signs right we must have $k > \kappa$. Since $\omega/\omega_0 \ll 1$ we then have $k \gg \kappa$ and $2\kappa k \approx \frac{\omega}{\omega_0}k^2$. Furthermore the $(\Delta\rho)$-terms give, with $\omega \ll \omega_0$ and $\kappa \ll k$, $\frac{\omega^2}{c_0^2} \approx k^2$. We get $\kappa = \frac{\omega k}{2\omega_0} = \frac{\omega^2}{2\omega_0 c_0}$. This lengthy discussion postponed the argument that $0 < \kappa \ll k$. The calculations are much shorter if $0 < \kappa \ll k$ is introduced as an early resonable assumption, and that can be enough for a full score.

5. Sliding on Water (9p)

A box is sliding on a layer of water. The box moves with a velocity $U(t)$. The water has thickness h. The water meets ground at $z = 0$. The ground is at rest.

The box moves in the x-direction, and we assume that the water is incompressible with velocity

$$\boldsymbol{v}(z,t) = (f(z,t),\, 0,\, 0).$$

a) [2p] Assume that water is a Newtonian fluid with constant viscosity. With gravity in the z direction and pressure depending only on z, simplify the Navier–Stokes equation as much as possible, and show that f must solve the diffusion equation, $\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial z^2}$. Relate the diffusion constant D to constants in the Navier–Stokes equation.

The advective term is $(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = f\frac{\partial}{\partial x}\boldsymbol{v}(z,t) = 0$, and we have $\nabla \cdot \boldsymbol{v} = 0$ from incompressibility or explicit calculations. Navier–Stokes reduces to

$$\frac{\partial \boldsymbol{v}}{\partial t} = \boldsymbol{g} - \frac{1}{\rho_0}\nabla p + \nu\nabla^2\boldsymbol{v}.$$

The y component gives trivially $0 = 0$. The z component gives $0 = -g_0 - \frac{1}{\rho_0}\frac{\partial p}{\partial z}$. The x component gives (using the stated properties of $\boldsymbol{g}$ and p) $\frac{\partial f}{\partial t} = \nu\frac{\partial^2 f}{\partial z^2}$. In the last step we have used that $\nabla^2 f(z,t) = \frac{\partial^2 f}{\partial z^2}$. This is the diffusion equation with $D = \nu$.

b) [2p] The box has a contact area S with the water, and will experience a drag force $F = S\sigma_{xz}$. Here, σ_{xz}, taken at $z = h$, is the shear force in the x-direction acting on the box bottom (whose surface is perpendicular to the z-direction). The box has finite mass M, and will be slowed down by the drag force, so that $M\frac{\mathrm{d}U}{\mathrm{d}t} = -F$. Show that f must solve

$$\frac{\partial f(h,t)}{\partial t} = -\alpha\frac{\partial f(h,t)}{\partial z}.$$

and determine the constant α. (Here,$\frac{\partial f(h,t)}{\partial z}$ means $\frac{\partial f}{\partial z}$, taken at $z = h$.)

We have $\sigma_{xz} = \eta\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right) = \eta\frac{\partial f}{\partial z}$. The no-slip boundary condition at top of the water (the box bottom) is $f(h,t) = U(t)$, so that $M\frac{\partial f(h,t)}{\partial t} = -S\eta\frac{\partial f(h,t)}{\partial z}$. Divide by M to get $\alpha = S\eta/M$.

c) [3p] Show that

$$f(z,t) = \exp(-s_k t)[A_k\cos(kz) + B_k\sin(kz)],$$

is a possible solution to the diffusion equation, and determine how s_k depends on k. Then, use boundary conditions to determine A_k, and to show that k must satisfy

$$kh\tan(kh) = \varepsilon.$$

Specify the dimensionless constant ε.

Apart from the value of ε, this problem can be solved independently of (b).
We get $\frac{\partial f}{\partial t} = -s_k f$ and $\nu\frac{\partial^2 f}{\partial z^2} = -\nu k^2 f$ which solves the diffusion equation with $s_k = \nu k^2$. Boundary condition for the floor at rest is $0 = f(0,t) = \exp(-s_k t)A_k$, so that $A_k = 0$. (Side note: since there are in the end many possible k, one might ask if this condition

should be relaxed to $\sum_{k_n} \exp(-\nu k_n^2 t) A_{(k_n)} = 0$, but since this should hold for every t, we get $A_k = 0$ for every k.)
Using the result in (b) (even if we failed to show it) gives $-s_k \exp(-s_k t) B_k \sin(kh) = -\alpha k \exp(-s_k t) B_k \cos(kh)$ so that $\tan(kh) = \alpha k/s_k = \alpha/(\nu k)$ and $kh \tan(kh) = \alpha h/\nu$. With α from (b) we get $\varepsilon = S\eta h/M\nu = S\rho_0 h/M$, where ρ_0 is the water density.

d) [1p] We can safely assume that the mass of the water under the box is much smaller than the mass of the box. Use this to motivate that $\varepsilon \ll 1$.

We have ε as the ratio of the water mass and box mass, and so according to the problem it is $\ll 1$.

e) [1p] One solution is then $k_0 \approx \frac{\sqrt{\varepsilon}}{h}$, (since $k_0 h \ll 1$ gives $\tan(k_0 h) \approx k_0 h$) but there are also solutions $k_n \approx n\pi/h$. Use the expression for s_k to motivate why only the solution with k_0 is interesting for large times t.

For this problem, we only need to know s_k from (c). Since the exponential dampening $\exp(-s_k t)$ is stronger for larger s, only the smallest s will matter at large times. With $s \propto k^2$ that implies the smallest k.