Exercises FYTA14, 2019. Some answers.

Appendix C

1. *Basic vector calculus operations*

a)

$$\begin{aligned} \nabla\phi &= (2xy,\, x^2,\, 1) \\ \nabla\phi &= (\cos x \cos y,\, -\sin x \sin y,\, 0) \end{aligned}$$

b)

$$\begin{aligned} \nabla \cdot \mathbf{u} &= 3x \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned}$$

c)

$$\begin{aligned} \nabla \times \mathbf{u} &= (2y - 3,\, 0,\, y) \\ \nabla \times \mathbf{u} &= (z \cos x \sin y,\, -z \sin x \cos y,\, \sin x \sin y + 2x \cos x \sin y) \end{aligned}$$

d) Show that.

e) Show that.

f) $\nabla^2 \mathbf{u} = (2,\, 0,\, 2)$, $\nabla(\nabla \cdot \mathbf{u}) = (3,\, 0,\, 0)$.

2. *Book C.1-C.3*

Show that.

3. *Book C.4*

C.4b: We must keep the parenthesis in $\nabla \cdot (\nabla \times \boldsymbol{V})$ and $\boldsymbol{V} \times (\nabla \times \boldsymbol{V})$. It is also safer to keep the parenthesis in $\nabla(\nabla \cdot V)$ and $(\nabla \cdot \nabla)\boldsymbol{V}$, though the book notation allows them to be omitted.

4. *Field Potentials*

$$\begin{aligned} &a) & \Phi &= xy + C, \\ &b) & \nabla \times \mathbf{F} &\neq (0,0,0), \\ &c) & \Phi &= \sin(xz) - xe^{-y} + C, \end{aligned}$$

5. *Find the pressure (5p)*

a) We must have $\nabla \times \nabla p = 0$, which gives $A = 2$. It is also possible to start solving (b), and find out which A is possible.

b) $p = R(\frac{1}{2}x^2 + 2zx + y^2 + yz - z^2 + D)$, where D is an undetermined constant.

6. *Important terms*

a)

$$\begin{aligned}
[(\boldsymbol{v} \cdot \nabla)\boldsymbol{v}]_x &= v_x \frac{\partial}{\partial x} v_x + v_y \frac{\partial}{\partial y} v_x + v_z \frac{\partial}{\partial z} v_x \\
[\nabla p]_x &= \frac{\partial p}{\partial x} \\
\left[\nabla^2 \boldsymbol{v}\right]_x &= \frac{\partial^2}{\partial x^2} v_x + \frac{\partial^2}{\partial y^2} v_x + \frac{\partial^2}{\partial z^2} v_x \\
[\nabla(\nabla \cdot \boldsymbol{v})]_x &= \frac{\partial}{\partial x}(\frac{\partial}{\partial x} v_x + \frac{\partial}{\partial y} v_y + \frac{\partial}{\partial z} v_z)
\end{aligned}$$

b)

$$\begin{aligned}
\tfrac{\partial}{\partial t} v_x &= \nu \tfrac{\partial^2}{\partial z^2} v_x \\
0 &= 0 \\
0 &= -g_0 - \tfrac{1}{\rho} \tfrac{\partial}{\partial z} p
\end{aligned}$$

Chapter 2

7. *Book 2.1, 2.4, 2.5, 2.8*

2.8 before a: Constant bulk modulus K and depth z gives $p = p_0 + K \ln \frac{\rho}{\rho_0}$ and $\frac{1}{\rho} = \frac{1}{\rho_0} - \frac{g_0}{K} z$.
2.8a) Partly "show that". Critical depth: $z = \frac{K}{\rho_0 g_0}$.

8. *Book 2.2*

a) Side i with water depth d_i: $p_i(z) = p_0 + \rho_0 g_0 (d_i - z)$ for $z \leq d_i$ and $p_i(z) = p_0$ for $z > d_i$.
b) ca $2.7 \cdot 10^6$ N.
c) ca 10^7 Nm.
d) 3.8 m (exactly).

9. ***The Standard Atmosphere (6p)***

a) $b = a/T_0$ and $c = g_0/(R_{\mathrm{air}}a)$.

b) Solving for ρ gives $\rho(z) = \rho_0(1 - bz)^{c-1}$. The numerical help in the problem gives $c \approx 5$ and $b \approx 1/(44\,\mathrm{km})$, with other values given, it gives $\rho(11\,\mathrm{km})/\rho(0) \sim (1-\frac{1}{4})^4 = (3/4)^4 \sim 0.3$.

c) $\gamma = c/(c-1)$

10. ***Relations between barotropic equations***

a) Incompressibility corresponds to $\gamma \to \infty$, isothermal model to $\gamma = 1$.

b) Show that. Hint: use $\gamma \to 1^+$ rather than $\gamma = 1$ for isothermal model

Chapter 3.1

11. ***Book 3.1, 3.2, 3.3***
3.2: ca $2\,\mathrm{cm}$

Chapter 12

12. ***Eq. of continuity: Sources and sinks***

a) $\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) = 0$.

b) Inverse time.

c) Show that.

d) Show that.

e) Show that.

f) Inversely proportional to the square of the distance.

13. ***Flow visualization (5p)***

a) With standard x and y axes, we get horizontal lines pointing right at $t = 0$, $45°$ pointing up+right at $t = \frac{1}{2}$ and vertical lines pointing up at $t = 1$.

b) The trajectory starts horizontally to the right at $(0, 0)$, bends upwards roughly as a circle, and ends vertically upwards in (1/2, 1/2).

c) The streak line starts horizontally to the left at (1/2, 1/2), bends downwards roughly as a circle, and ends vertically downwards in $(0, 0)$.

14. ***Book 12.2, 12.3, 12.4***
12.2:
a) Parallel lines, in the time-varying direction $(cos\omega t, \sin \omega t)$.
b) Counter-clockwise circle with radius a/ω.
c) Clockwise circle with radius a/ω.

12.4: The velocity in the 1 inch pipe is 27/32 of the velocity in the thick (3/4 inch) branch, and 3/4 of the velocity in the thin (1/2 inch) branch.

15. ***Book 12.6, 12.7***

Chapter 18.0-18.2

16. ***Geostrophic wind (calculator needed)***

a) TBA

b) Show that. (Note, with the precision given, both values should be $0.002\,\mathrm{Pa/m}$.)

c) With $|\partial_x p| = |\partial_y p|$, it is a south-western wind (coming from south-west, velocity vector pointing north-east). Components $v_x = v_y \sim 14\,\mathrm{m/s}$ (estimated without calculator).

17. ***Isobar surfaces (6p)***

a) $C = -\frac{1}{g_0} 2\Omega_z v_0$.

b) The gradient is perpendicular to contour surfaces, so $\nabla p \propto \widehat{\mathbf{e}}_z - C\widehat{\mathbf{e}}_y$ is normal to isobar surfaces, with C defined above, independently of any variation in ρ. As a side note, this problem is restricted to the situation with constant $\mathbf{v}$, which in principle restricts us to barotropic relations $p = p(\rho)$.

$\boldsymbol{c}$) With $2v_0/g_0 \approx 8\,\mathrm{s}$, flying distance $\Delta y = -350 \cdot 10^3\,\mathrm{m}$, and altitude slightly above 45° north, so that cosine of this angle is a bit more than $\cos\pi/4 = 1/\sqrt{2}$ (shall we say 0.8?), we get $\Omega_z = 0.8 \cdot 2\pi/(24 \cdot 3600)\,\mathrm{s}^{-1} \approx 0.8/(4 \cdot 3600)\,\mathrm{s}^{-1} = (1/18000)\,\mathrm{s}^{-1}$ and $\Delta z = (8/18000) \cdot 350000\,\mathrm{m} \approx 160\,\mathrm{m}$. The final answer can be quite far away, a factor 5 or so, and still earn full score, if the used rough estimates are motivated.

18. ***Missing data (6p)***

a) With ρ the density, $\mathbf{\Omega}_z$ the local (vertical) angular velocity of the earth, $v_0 = 6\,\mathrm{m/s}$ and $L = 20\,\mathrm{km}$, the missing data is $p_B = p_A - L\rho\Omega_z v_0$.

b) In the southern hemisphere, Ω_z is negative, so $p_B > p_A$.

c) A typical length scale is $S = 500\,\mathrm{km}$ and our only available estimate of a typical velocity is v_0. The Rossby number at an angle θ from a pole of the earth is $v_0/(2S\Omega|\cos\theta|) \sim 6/(2 \cdot 5 \cdot 10^5 \cdot 0.5 \cdot 10^{-4}|\cos\theta|) = 6/(50|\cos\theta|)$. “Near” the equator, meaning $|\cos\theta| < 1/8$ or so, the Rossby number is too large for geostrophic balance to be a valid assumption. (A more trivial remark is that point A must be at least 20 km away from the south pole for point B to exist…)

19. ***Book 18.2 with hints***
The interface is 1-2 m higher on the western bank, depending on how you interprete “4%” and “25%”. Compared to the example on page 314, the interface difference is about 5 times as big as the upper surface shift, and in the opposite direction.

20. ***Extra-curricular: Check section 18.5***

Chapter 13

21. ***Book 13.2, 13.3, 13.4***
13.2: ca 1 min 4 s.

22. ***Ideal flows (6p)***

a) Only case (iii) is steady.

b) Case (i) and (iii) are irrotational.

c) Case (iii) could be the velocity field described. The pressure is $p = C - (x^2 + y^2)$ where C is a constant.

23. ***Pitot by-pass (6p)***
$U \approx 2\,\mathrm{m/s}$.

24. ***Velocity Measurement (6p)***

a) The surface will be higher in the pipe attached to the wide region.

b) The height difference is $\frac{1}{20}\left(\frac{1}{0.8^2} - 1\right)\,\mathrm{m} \sim 3\,\mathrm{cm}$.

Chapter 14

25. ***Book 14.1,14.3,14.5***

Chapter 25

26. ***Book 25.3***
The group velocity scales as the square root of the depth (near beach we have shallow-water waves), so the more distant part of a wave travels faster, causing the wave to bend towards the shore, arriving more or less parallell.

27. ***How to measure a water wave***

a) Show that

b) Show that

c) show that

d) About 0.97 for the tsunami and very roughly 10^{-13} for the smaller wave. The conclusion is that the correction can be neglected for shallow water waves. Note that the tsunami is a shallow-water wave, despite being out in the ocean. That is because the ocean is shallow compared to the wavelength.

28. *Deep-water Waves (7p)*

a) Show that

b) In part show that. At $z = h$ we want $\frac{\partial v_z}{\partial t} = \frac{\partial^2 h}{\partial t^2}$, corresponding to $-a\omega^2 \exp(kh)\cos(kx - \omega t) = -\omega^2 a \cos(kx - \omega t)$. This is satisfied if $\exp(kh) \approx 1$, which introduces the condition $kh \ll 1$. (For deep-water waves, the other condition $a \ll d$ is trivially satisfied.)

c) Show that.

Chapter 15

29. *Book 15.1, 15.4*

30. *Frictionless circulation*

a) $V_{12} = \omega$, $V_{21} = -\omega$. All other $V_{ij} = 0$.

b) Show that.

31. *A pipe*

a) Show that.

b) $0 = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right)$, $0 = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}$, $0 = -g_0 - \frac{1}{\rho_0}\frac{\partial p}{\partial z}$.

c) Show that.

d) Show that.

e) Dimension of G is that of $\nabla^2 \mathbf{v}$, so 1 over length and time. Since G depends only on different length scales and $\langle v_x \rangle$, the average velocity is the only variable that can give time dimension in the denominator. To do so, G must be linear in $\langle v_x \rangle$.

32. ***Shear wave***

33. ***Sound attenuation***

34. ***Book 15.6, revised and with hints***

Chapter 18.3

35. ***Ekman layer: Check equation 18.27***

a) Show that.

b) Show that. Do not forget the boundary $z \to \infty$

c) Show that.

36. ***Exam 2016-06-03, problem 3 "Barotropic, Geostrophic Catastrophe"***

a) The chain rule $\frac{1}{\rho}\nabla p = \omega'(p)\nabla p = \nabla\omega$ and the gravitational potential rewrites the geostrophic balance to $0 = -\nabla\Phi - \nabla\omega - 2\mathbf{\Omega}_z \times \boldsymbol{v}$. Taking the curl of this, and using $\nabla \times \nabla = 0$ we get the result.

b) First x component: $0 = [\nabla \times (\hat{\mathbf{e}}_z \times \boldsymbol{v})]_x = \frac{\partial}{\partial y}[\hat{\mathbf{e}}_z \times \boldsymbol{v}]_z - \frac{\partial}{\partial z}[\hat{\mathbf{e}}_z \times \boldsymbol{v}]_y$. Since a cross product involving $\hat{\mathbf{e}}_z$ is perpendicular to the z axis, the first term vanishes, and we get $0 = -\frac{\partial}{\partial z}[\hat{\mathbf{e}}_z \times \boldsymbol{v}]_y = -\frac{\partial}{\partial z}(v_x - 0)$ so $\frac{\partial}{\partial z}v_x = 0$. Similarly, the y component of the full epxression gives $\frac{\partial}{\partial z}v_y = 0$.

c) Most important is to consider friction (or viscosity), which creates a boundary layer interpolating between ground and a region of geostrophic balance. The key "qualitative" way it affects the wind is that it creates a component of the wind going (on average, in case of turbulence) in the direction of the pressure gradient force.

The topics above were considered enough for a full score. If some of it was lacking, points could be rewarded for more subtle information, *e.g.*: if the simple assumptions of Ekman holds, the wind as a function of height will "spiral"; to explain data, we must include the contribution of turbulence on the effective viscosity.