The tasks are not sorted in order of difficulty.
Read the text carefully before you start to solve a problem.
Present partial results, even if your solution is incomplete.
Many sub-problems can be solved independently of previous sub-problems.
Cartesian coordinates in an inertial frame are used, unless a rotating system is explicitly referred to.

1. The Standard Atmosphere (6p)
The temperature of the atmosphere at height $z$ above ground is roughly

$$T(z) = T_0 - az,$$

where $T_0$ is the ground temperature and $a$ is a parameter. With $a = 6.5 \text{ K/km}$, this parametrization works well for the lowest 11 km of the atmosphere. The air is assumed to obey the ideal gas law, which can be written

$$p = \rho R_{\text{air}} T.$$

The value of $R_{\text{air}}$ (the “specific gas constant” for air) is $287 \text{ m}^2/(\text{s}^2 \text{ K})$.

a) [3p] Assume hydrostatic equilibrium and constant gravitational acceleration $g_0 \approx 10 \text{ m/s}^2$. Let $p_0$ and $\rho_0$ be the pressure and density at ground, respectively. Show that $p(z)$ is of the form

$$p(z) = p_0(1 - bz)^c$$

and find expressions for $b$ and $c$.

b) [1p] Show that the atmospheric model is polytropic, $p = C \rho^\gamma$, where $C$ and $\gamma$ are constants. Find an expression for $\gamma$.

c) [2p] Suppose the ground temperature is $T_0 = 286 \text{ K} \approx 13^\circ \text{C}$. Make a rough numerical estimate of the density at height 11 km, relative to ground density – in other words, estimate the ratio $\rho(11 \text{ km})/\rho_0$. Hint: $286/6.5=44$. You may use the estimate $g_0/R_{\text{air}}a \approx 5$. 

2. Pressure in a tornado (10p)
The “Rankine vortex” can be used as a simple model for tornadoes. It’s defined by
\[ \mathbf{v} = (\Omega, -\Omega y, 0), \]
where the angular frequency \( \Omega \) depends on the horizontal radius \( r = \sqrt{x^2 + y^2} \) as
\[ \Omega(r) = \begin{cases} \Omega_0, & 0 \leq r \leq R, \\ \Omega_0 \frac{R^2}{r^2}, & r > R. \end{cases} \]
\( R > 0 \) and \( \Omega_0 \) are constants. The region \( 0 \leq r < R \) is called the core of the vortex.
First, we look in the core,
\[ \mathbf{v}^{(\text{core})} = (-\Omega_0 y, \Omega_0 x, 0). \]

a) [1p] Show that the vorticity in the core is a non-zero constant.
b) [1p] Calculate the stream function in the core and show that the stream lines are circles.
c) [2p] Assume that \( \mathbf{v}^{(\text{core})} \) describes the flow of an ideal, incompressible fluid with constant density \( \rho_0 \) and zero gravity. Calculate the pressure in the core. The answer should contain one free constant.

Next, we look outside the core, where the velocity is
\[ \mathbf{v}^{(\text{out})} = R^2 \left(- \frac{\Omega_0 y}{x^2 + y^2}, \frac{\Omega_0 x}{x^2 + y^2}, 0\right) = \frac{R^2}{x^2 + y^2} (-\Omega_0 y, \Omega_0 x, 0). \]

d) [1p] Show that the vorticity outside the core is zero.
e) [1p] Calculate the stream function outside the core and show that the stream lines are circles.
f) [1p] For future convenience, show that
\[ \mathbf{v}^{(\text{out})} \cdot \nabla \frac{R^2}{r^2} = 0. \]
(This result is useful together with the product rule for derivatives. Obviously, the result may be used below even if you failed to show it.)
g) [2p] Calculate the pressure outside the core (under the same assumptions as in the core). The answer should contain one free constant.

Finally, we combine:

h) [1p] Assuming that the pressure is continuous at \( r = R \), show that the difference \( \Delta p \) in pressure between \( r = 0 \) and \( r = \infty \) is \( -\rho_0 \Omega^2 R^2 \).

Comment: this explains the very low pressure in the centre of a tornado!
3. Half full or half empty (8p)
A large, open tank with constant cross-section $A_0$ and height $h$ is filled with water. At time $t = 0$, an outlet with cross-section $A < A_0$ is opened at the bottom, and water exits the outlet with a velocity $v$, while the surface in the tank sinks with a velocity $v_0$.

a) [1p] Assuming constant density, find a relation between $v$ and $v_0$, independent of $h$.

b) [1p] Adding the assumptions of steady, ideal flow and constant gravity, find a relation between $v$, $v_0$ and $h$.

c) [1p] Combining the above, express the velocity through the outlet as a function of $h$.

d) [3p] Calculate the ratio $T_{1/2}/T$, where $T_{1/2}$ is the time required for the tank to empty halfway and $T$ the time required for the tank to empty completely.

e) [2p] Hopefully, your result in (c) is un-physical in the limit $A \to A_0$ (representing a situation where almost the whole bottom of the tank is opened as an outlet). What assumption(s) above would be unreasonable for large $A$?

4. Missing data (6p)

A weather station (or balloon, or ship) measures the wind to be 6 m/s, pointing 30° west of the north direction. The pressure is measured to be 100.5 kPa. From another position, 20 km south of the first, the wind is the same but pressure data is missing.

a) [2p] Assuming geostrophic balance and a constant pressure gradient between the measurement positions, find an expression for the missing pressure data. Define all variables introduced in the expression. (Reminder: $\sin(30°) = 1/2, \cos(30°) = \sqrt{3}/2$.)

b) [1p] If the positions are in the southern hemisphere, would the result for the missing pressure data be higher or lower than 100.5 kPa?

c) [3p] Assume that the wind conditions are fairly similar over distances of about 500 km (remarkably steady weather, this is). Discuss in what regions on our planet that our approach to find the missing pressure would be misleading! The answer need not be very precise (the word “misleading” is not well defined), but should be motivated with estimates and calculations. You may set the earth’s angular velocity to $0.5 \cdot 10^{-4}$ s$^{-1}$ and assume a constant air density 1.2 kg/m$^3$. 