The tasks are not sorted in order of difficulty.
Read the text carefully before you start to solve a problem.
Present partial results, even if your solution is incomplete.
Many sub-problems can be solved independently of previous sub-problems.
Cartesian coordinates in an inertial frame are used, unless a rotating system is explicitly referred to.

1. Ideal flows (6p)

Below are three two-dimensional velocity fields given in non-dimensional variables.

\[ i) \quad \mathbf{v} = (t \sin x, 0) \]
\[ ii) \quad \mathbf{v} = (y + t, 2) \]
\[ iii) \quad \mathbf{v} = (x, -y) \]

a) [1p] For each one of them, determine if it represents steady flow or not.
b) [2p] For each one of them, determine if it represents irrotational flow or not.
c) [3p] Could any of them be the velocity field for an incompressible ideal flow in the domain \( y > 0 \), bounded from below by a stationary wall at \( y = 0 \)? If so, compute the corresponding pressure at zero gravity, with the constant non-dimensional density \( \rho_0 = 2 \).

2. Find the pressure (5p)

A group of students solve a complicated problem in fluid dynamics. Everyone agrees that the pressure \( p(x, y, z) \) must fulfil

\[ \nabla p = R(x + 2z, 2y + z, Ax + y - 2z). \]

Here, \( R \) is a constant of appropriate dimension, and \( A \) is a dimensionless constant. Based on other parts of the problem, one student has found \( A = 1 \), while another student thinks that \( A = 2 \).

a) [2p] Is any of the suggested values of \( A \) correct? If so, which one? (An unmotivated guess will not reward any points.)
b) [3p] Find the most general expression for \( p(x, y, z) \), given the correct value of \( A \).
3. Canal flow (8p)

A canal has vertical walls and constant width. It is directed in the x-direction and the vertical is in the z direction. The gravitational field is thus \( \mathbf{g} = (0, 0, -g_0) \) with \( g_0 \approx 10 \text{ m/s}^2 \).

Assume that water in the canal obeys steady, ideal, incompressible flow, and that the dynamics is independent of the position \( y \) across the canal.

In an upstream region, the bottom is flat at \( z = 0 \) and the surface is flat at \( z = s_0 \). The velocity in this region is uniform, \( \mathbf{v} = (v_0, 0, 0) \). A bit downstream, however, the bottom follows a curve \( b(x) \), and the surface is then described by a function \( s(x) \). For simplicity, assume that \( v_x \) in this region is independent of \( z \), so that

\[
\mathbf{v} = (v_x(x), 0, v_z(x, z)).
\]

a) [1p] Show that the velocity at the surface satisfies \( v_z(x, s) = s'(x)v_x(x) \).

(You will need this result below, and may of course use it even if you failed to show it.)

b) [4p] Assuming constant air pressure, show that the functions \( s(x) \) and \( b(x) \) must satisfy

\[
(1 + s'^2)C \frac{s_0^2}{(s - b)^2} + s = C + s_0. \tag{1}
\]

Express \( C \) in the constants provided in the problem description.

**Hint:** Follow a streamline at the surface. (And again, you may use this result below, even if you failed to show it.)

c) [3p] Suppose the surface has negligible curvature, i.e., \( s''(x) \approx 0 \). With \( s_0 \approx 1 \text{ m}, v_0 \approx 1 \text{ m/s}, \) and \( s \approx 1 \text{ m}, \) will the surface and bottom have opposite slopes or not? Opposite slopes means that the surface gets lower (\( ds < 0 \)) over a bump at the bottom (\( db > 0 \)). An unmotivated guess will not reward any points.

**Hint:** The relation in eq. (1) can be used to write \( b \) as a function of \( s \) and \( s' \). With \( s'' \approx 0 \), you can treat \( s' \) as a constant, and calculate \( db/ds \) to find the relative signs of the slopes. If you lack an expression for \( C \), you can solve most of the problem and state a partial result as a condition on \( C \).
4. Isobar surfaces  (6p)

On a weather map, isobars are lines, representing constant pressure for fixed height (near ground). Adding the vertical dimension, we get surfaces of constant pressure.

a) [3p] For an incompressible fluid, show that constant pressure in a region with a geostrophic flow \( \mathbf{v} = (v_0, 0, 0) \) is described by a surface \( z = Cy + B \), for all \( x \). Express \( C \) in \( v_0 \) and the physical constants appearing in the equation for geostrophic balance.

b) [1p] Show that the result in (a) does not require constant density, and in fact is valid under geostrophic balance, independently of any particular relation between pressure \( p \) and density \( \rho \).

c) [2p] An airplane flies above Lund at height 980 m (where we assume geostrophic balance to hold). It measures strict western wind of 40 m/s (western wind means that \( \mathbf{v} \) points east), and a pressure \( p_L \). Later, the airplane is near Berlin, 350 km straight south of Lund. Assuming constant wind during the whole trip (we do not worry too much about realism here...), at what altitude would the airplane measure the same pressure? Make rough numerical estimates when needed.

5. Turbulence  (5p)

a) [1p] What is an eddy?

b) [4p] How can the cascade model of eddies and its impact on the kinematic viscosity explain that the boundary layer around an object flying through air becomes thinner when turbulence sets in? Your explanation should not be too short, but may be quite qualitative (no need for exact formulas or precise numerical values).