Exam, FYTA14, 2015-08-25, 9.15-15.15

Allowed material: One a4 sheet with notes, writing material.

30 points total, 15 points to pass, 24 points for distinction.

The tasks are *not* sorted in order of difficulty.
Read the text *carefully* before you start to solve a problem.
Present partial results, even if your solution is incomplete.
Many sub-problems can be solved independently of previous sub-problems.
Cartesian coordinates in an inertial frame are used, unless a rotating system is explicitly referred to.

1. Sound Waves (7p)

Consider an ideal, compressible gas close to an equilibrium where the velocity field is $\boldsymbol{v}_0 = 0$ and density ρ_0 and pressure p_0 are constants. Let $\boldsymbol{v}$, $\Delta\rho$ and Δp denote the deviations from the equilibrium values, so that $\rho = \rho_0 + \Delta\rho$, etc. Then, the linearized Euler equations become

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\frac{1}{\rho_0}\nabla(\Delta p) \tag{1}$$

and the linearized equation of contiuity becomes

$$\frac{\partial \Delta\rho}{\partial t} = -\rho_0 \nabla \cdot \boldsymbol{v}. \tag{2}$$

a) [3p] Suppose the flow is isentropic. Show that $\Delta\rho$ satisfies the wave equation

$$\frac{\partial^2 \Delta\rho}{\partial t^2} = c_0^2 \nabla^2(\Delta\rho), \tag{3}$$

where $c_0^2 = \frac{\gamma p_0}{\rho_0}$, and $\gamma = c_p/c_v$ is the adiabatic index of the gas.

b) [2p] Show that the wave equation eq. (3) is satisfied by a plane wave

$$\Delta\rho = \rho_1 \sin(kx - \omega t + \phi_1),$$

where ρ_1, ϕ_1, k and ω are constants, provided ω and k satisfies a necessary relation. Give that relation.

c) [1p] Show that the Euler equations linearize to eq. (1).

d) [1p] Show that the equation of continuity linearizes to eq. (2).

2. Ideal Flows (6p)

Below are three 3-dimensional velocity fields given in non-dimensional variables:

$$
\begin{aligned}
i) \quad & \boldsymbol{v} = (tx, -ty, t), \\
ii) \quad & \boldsymbol{v} = (xy + xz, -yz, 1 - yz), \\
iii) \quad & \boldsymbol{v} = (\sin z, \cos z, 0).
\end{aligned} \tag{4}
$$

a) [1p] For each one of them, determine if it represents steady flow or not.

b) [2p] For each one of them, determine if it represents irrotational flow or not.

c) [1p] For each one of them, determine if it can represent incompressible flow with constant density.

d) [1p] Suppose the velocity field should represent a flow in the domain $z > 0$, bounded from below by a stationary wall at $z = 0$. ("The flow of an ideal fluid on a fixed floor at $z = 0$.") What boundary condition(s) must the velocity field satisfy?

e) [1p] Indeed, one field above could be the velocity field for an incompressible ideal flow satisfying the constraints in (d). Show that the corresponding pressure is constant at zero gravity.

3. Betweem Two Planes (6p) Consider an incompressible, viscous fluid bounded by two planes perpendicular to the z-axis, so that $0 < z < d$. Assume steady flow in the x direction, depending only on the height above the lower plane: $\boldsymbol{v} = (v_x(z),\ 0,\ 0)$.

Assume constant density ρ_0, a pressure independent of x and y (in other words, $p = p(z)$), and a gravitational acceleration in the z direction $\mathbf{g} = (0, 0, -g_0)$.

a) [3p] Reduce Navier–Stokes equations, to get as simple equations as possible for each of its x-, y-, and z-components. In particular, verify that pressure obeys the euqation for hydrostatic equilibrium, and that the velocity satisfies

$$v_x''(z) = 0.$$

b) [1p] The lower plane, at $z = 0$ is lying still, while the upper plane, at $z = d$, has constant velocity $\boldsymbol{U} = (U_0,\ 0,\ 0)$. Find the solution to v_x satisfying boundary conditions.

c) [2p] The fluid will act on the planes with a shear force along the x-direction. Show that this "drag" leads to a force D per area A which has magnitude

$$\frac{D}{A} = \frac{1}{d}\rho\nu U_0.$$

4. Geostrophic Wind (5p)

Four weather-stations (A, B, C and D) are located in the corners of a square with side 20 km. Station A and B are 20 km south of C and D. Station A and C are 20 km west of B and D.

Simultaneous pressure measurements are $p_A = 1002.0\,\text{hPa}$ ($1\,\text{hPa} = 100\,\text{Pa}$), $p_B = 1002.2\,\text{hPa}$, $p_C = 1002.1\,\text{hPa}$ and $p_D = 1002.3\,\text{hPa}$.

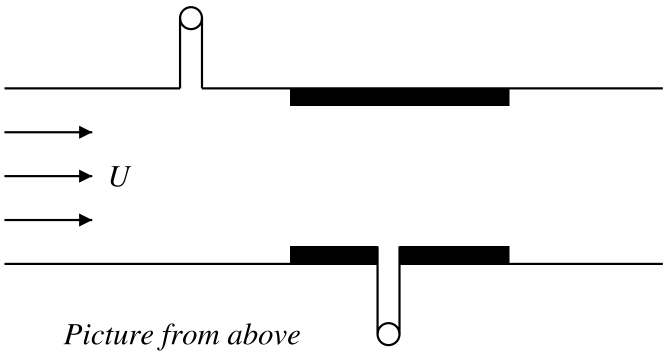
Assume geostrophic balance and a constant pressure gradient.

a) [2p] Draw a sketch illustrating the direction of the geostrophic wind! The stations are in the northern hemisphere.

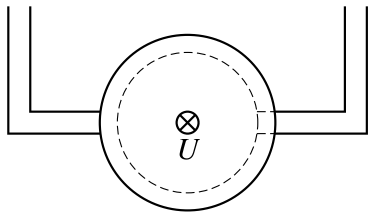
The region is at 45° north, and the air density is $1.2\,\text{kg/m}^3$.

b) [3p] Approximately, what is the magnitude of the wind?

5. Velocity Measurement (6p)



Picture from above



Picture from inlet

Consider steady, ideal water flow filling a cylindrical pipe. The pipe radius is mostly constant, but in one region it is 5% smaller.

The water also enters two side tubes, attached on the horizontal diameter of the pipe. The tubes bend upwards, are open on top and in contact with air. Air pressure is assumed to be constant.

In the picture, the water surfaces in the tubes are not drawn. Since the tubes are connected to regions of different radius, one surface is higher than the other.

a) [1p] Which of the two side tubes has the higher water surface? Motivate your answer.

b) [5p] The height difference is 5 mm. What is approximately the velocity U in the wider region? You may use the gravitational acceleration $\approx 10\,\text{m/s}$. *Hint:* The main task is to find an expression relating the velocity to other quantities stated in the problem. For a numerical result, do not hesitate to make approximations like $(1+x)^k \approx 1 + kx$ for reasonably small x.