Exam, FYTA14, 2016-06-03, 10.15-15.15

Allowed material: One a4 sheet with notes, writing material.

30 points total, 15 points to pass, 24 points for distinction.

The tasks are *not* sorted in order of difficulty.
Read the text *carefully* before you start to solve a problem.
Present partial results, even if your solution is incomplete.
Many sub-problems can be solved independently of previous sub-problems.
Cartesian coordinates in an inertial frame are used, unless a rotating system is explicitly referred to.

1. Deep-water Waves (7p)

Consider water with constant density ρ_0 and in constant gravity $\boldsymbol{g} = (0,\, 0,\, -g_0)$. The water is at rest with its surface at $z = 0$ amd a large depth d. We add a perturbation which creates a small amplitude surface wave, so that the surface is at

$$h(x,t) = a\cos(kx - \omega t).$$

If a is small enough, we can neglect the advective term in Euler's equations, so that the velocity is determined by

$$\frac{\partial \boldsymbol{v}}{\partial t} = \boldsymbol{g} - \frac{1}{\rho_0}\nabla p. \tag{1}$$

a) [2p] Show that constant density and eq. (1) implies that

$$\Phi^*(x,z,t) = g_0 z + \frac{1}{\rho_0}p(x,z,t)$$

satisfies the Laplace equation, $\nabla^2\Phi^* = 0$.

b) [3p] Show that

$$\Phi^* = \frac{a\omega^2}{k}\exp(kz)\cos(kx - \omega t) + C \tag{2}$$

is a solution. Find $\frac{\partial v_z}{\partial t}$ (or if you prefer, v_z) and show that it approximately satisfies the boundary conditions for $z \to -\infty$ and $z = h$. What are the constraints needed on the amplitude a for the approximations at $z = h$ to be valid?

c) [2p] Assume constant pressure p_0 at the surface and show, under the same approximations, the dispersion relation

$$\omega^2 = g_0 k.$$

2. Steady Flows (6p)

Below are three velocity fields for steady, ideal flows, given in non-dimensional variables:

$$\begin{aligned} i) \qquad & \boldsymbol{v} = (x,\, -y,\, 0), \\ ii) \qquad & \boldsymbol{v} = (-y,\, x,\, 0), \\ iii) \qquad & \boldsymbol{v} = (xy,\, yz,\, 0). \end{aligned} \tag{3}$$

a) [3p] Determine which flows that are incompressible, and which flows that are irrotational.

b) [1p] For one of the flows the Bernoulli field H is a global constant. Determine which one.

c) [2p] One of the velocity fields can be represented by a velocity potential Ψ. Determine which one, and find an expression for Ψ.

3. Barotropic, Geostrophic Catastrophe (6p)

For a barotropic atmosphere, we can define a pressure potential $\omega(p)$ obeying $\omega'(p) = \frac{1}{\rho(p)}$.

a) [2p] Use the pressure potential and $\boldsymbol{g} = -\nabla\Phi$, to show that the velocity field for a barotropic atmosphere in geostrophic balance satisfies

$$\nabla \times (2\boldsymbol{\Omega}_z \times \boldsymbol{v}) = 0.$$

b) [1p] Assume $\boldsymbol{\Omega}_z$ to be constant and pointing in the vertical direction $\hat{\mathbf{e}}_z$. Look at the separate components of the equation above and show that

$$\frac{\partial v_x}{\partial z} = 0, \qquad \frac{\partial v_y}{\partial z} = 0.$$

c) [3p] According to (b), the horizontal wind does not change with height! The no-slip condition at ground therefore implies that the horizontal wind must be zero for all z. Thus, winds cannot exist...

Obviously, our initial assumptions have been stretched too far. Discuss what effects near ground that have been neglected in the equation for geostrophic balance, and in qualitative terms how they affect the wind.

4. Sliding on Water (11p)

A box is sliding on a layer of water. The box moves with a velocity $U(t)$. The water has thickness h. The water meets ground at $z = 0$. The ground is at rest.

The box moves in the x-direction, and we assume that the flow in the water is described by

$$\boldsymbol{v}(z,t) = (f(z,t),\, 0,\, 0).$$

a) [1p] Show that this is consistent with a constant water density ρ_0.

b) [2p] Assume that water is a Newtonian fluid with constant viscosity, so that the flow obeys the Navier–Stokes equations. With gravity in the z direction and pressure depending only on z, simplify the Navier–Stokes equations as much as possible, and show that f must solve the diffusion equation, $\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial z^2}$. Relate the diffusion constant D to constants in Navier–Stokes equations.

c) [2p] The box has a contact area S with the water, and will experience a drag force $F = S\sigma_{xz}$. Here, σ_{xz}, taken at $z = h$, is the shear force in the x-direction acting on the box bottom (whose surface is perpendicular to the z-direction). The box has finite mass M, and will be slowed down by the drag force, so that $M\frac{\mathrm{d}U}{\mathrm{d}t} = -F$. Show that f must solve

$$\frac{\partial f(h,t)}{\partial t} = -\alpha\frac{\partial f(h,t)}{\partial z}.$$

and determine the constant α. (Here, $\frac{\partial f(h,t)}{\partial z}$ means $\frac{\partial f}{\partial z}$, taken at $z = h$.)

d) [3p] Show that

$$f(z,t) = \exp(-s_k t)[A_k\cos(kz) + B_k\sin(kz)],$$

is a possible solution to the diffusion equation, and determine how s_k depends on k. Then, use boundary conditions to determine A_k, and to show that k must satisfy

$$kh\tan(kh) = \varepsilon.$$

Specify the dimensionless constant ε.

e) [1p] We can safely assume that the mass of the water under the box is much smaller than the mass of the box. Use this to motivate that $\varepsilon \ll 1$.

f) [1p] One solution is then $k_0 \approx \frac{\sqrt{\varepsilon}}{h}$, (since $k_0h \ll 1$ gives $\tan(k_0h) \approx k_0h$) but there are also solutions $k_n \approx n\pi/h$. Use the expression for s_k to motivate why only the solution with k_0 is interesting for large times t.

g) [1p] To determine the final unknown konstants B_k, we must look at initial conditions at $t = 0$. Assume $U(t = 0) = U_0$. Furthermore, assume that the initial condition for the water is such that all $B_{(k_n)}$ are zero, except for k_0. With the approximation $\sin(k_0z) \approx k_0z$, determine the constant $B_{(k_0)}$.