Exam, FYTA14, 2016-08-26, 10.15-15.15

Allowed material: One a4 sheet with notes, writing material.

30 points total, 15 points to pass, 24 points for distinction.

The tasks are *not* sorted in order of difficulty.
Read the text *carefully* before you start to solve a problem.
Present partial results, even if your solution is incomplete.
Many sub-problems can be solved independently of previous sub-problems.
Cartesian coordinates in an inertial frame are used, unless a rotating system is explicitly referred to.

1. Velocity Fields (6p)

Below are three velocity fields, given in non-dimensional variables:

$$\begin{aligned} i) \quad & \boldsymbol{v} = (2xt,\, -2yt,\, t), \\ ii) \quad & \boldsymbol{v} = (2\sin y,\, 2\cos x,\, 0), \\ iii) \quad & \boldsymbol{v} = (2z - z^2,\, 0,\, 0). \end{aligned} \tag{1}$$

a) [1p] For each flow, determine if it is incompressible.

b) [1p] For each flow, determine if it is irrotational.

c) [1p] Determine which fields that are consistent with ideal flow above a floor at $z = 0$.

d) [3p] One of the fields is consistent with a steady, viscous flow above a floor at rest at $z = 0$. Determine which one. Show that the field solves the Navier–Stokes equations if pressure p and density ρ satisfies $\nabla p = \rho \boldsymbol{g} - 2\eta \widehat{\mathbf{e}}_x$, where $\boldsymbol{g}$ is the gravitational acceleration and η is the (constant) viscosity of the fluid.

2. Stream lines (5p)

Between time $t = 0$ and $t = 1$ (in non-dimensional variables), a 2-dimensional flow is described by

$$\boldsymbol{v} = (1 - t,\, t).$$

a) [1p] Scetch stream lines at $t = 0$, $t = \frac{1}{2}$ and $t = 1$.

b) [2p] Scetch (roughly) a particle trajectory from $t = 0$ to $t = 1$, starting at $(x,\, y) = (0,\, 0)$. What is the endpoint at $t = 1$?

c) [2p] Scetch (roughly) a streak line from $t = 0$ to $t = 1$, going through $(x,\, y) = (0,\, 0)$ at $t = 1$.

3. Pitot Tube (5p)

A thin, “L-shaped” tube with open ends is lowered into a canal. The opening under water is turned towards the flow in the canal. The water level inside the tube is then observed to be higher than the canal surface. Starting with ideal flow and other reasonable assumptions, derive an expression that relates the height difference h to flow velocity U in the canal.

4. Origin of Geostrophic Flow (6p)

Consider ideal flow expressed in a coordinate frame rotating with constant angular velocity $\mathbf{\Omega}$ with respect to an inertial frame.

a) [1p] What terms are then added to Euler’s equations for ideal flow?

b) [2p] In many earth applications, the $\mathbf{\Omega}$-dependent terms can be neglected or simplified. Describe how and give motivations.

c) [1p] What is the equation for geostrophic flow?

d) [2p] Let the flow have a typical velocity U and range over a typical length-scale L. How do you estimate if geostrophic flow is a valid approximation?

5. Dampened Sound (8p)

Consider an ideal, compressible gas close to an equilibrium where the velocity field is $\boldsymbol{v}_0 = 0$ and density ρ_0 and pressure p_0 are constants. Let $\boldsymbol{v}$, $\Delta\rho$ and Δp denote the deviations from the equilibrium values, so that $\rho = \rho_0 + \Delta\rho$, etc. Then, the linearized Euler equations become

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\frac{1}{\rho_0}\nabla(\Delta p) \tag{2}$$

and the linearized equation of continuity becomes

$$\frac{\partial \Delta\rho}{\partial t} = -\rho_0 \nabla \cdot \boldsymbol{v}. \tag{3}$$

a) [3p] Suppose the flow obeys a barotropic relation $p = p(\rho)$. Show that $\Delta\rho$ satisfies the wave equation

$$\frac{\partial^2}{\partial t^2}(\Delta\rho) = c_0^2 \nabla^2(\Delta\rho), \tag{4}$$

and describe how c_0 can be determined from the barotropic relation.

b) [1p] Show that the equation of continuity indeed linearizes to eq. (3).

c) [3p] Instead consider a viscous fluid, with constant viscosity η and bulk viscosity ζ. The vicuous terms in the the Navier–Stokes equations are linear in $\boldsymbol{v}$, and can immediately be added to the linearized Euler equations eq. (2). Show that this modifies the wave equation to

$$\frac{\partial^2}{\partial t^2}(\Delta\rho) = c_0^2 \left[\nabla^2(\Delta\rho) + \frac{1}{\omega_0}\nabla^2 \frac{\partial}{\partial t}(\Delta\rho)\right], \tag{5}$$

and find an expression for the introduced constant ω_0.

d) [1p] A possible solution to eq. (5) is a dampened sound wave,

$$\Delta\rho = \rho_1 \exp(-\kappa x)\cos(kx - \omega t), \tag{6}$$

where ρ_1, κ, k and ω are constants. In the limit of $\omega \ll \omega_0$, determine how the dampening coefficient κ depends on sound frequency ω.