

Exam, FYTA14, 2017-06-02, 10.15-15.15

Allowed material: One a4 sheet with notes, writing material.

30 points total, 15 points to pass, 24 points for distinction.

The tasks are *not* sorted in order of difficulty.

(They are instead sorted in order of kitchen water flow.)

Read the text *carefully* before you start to solve a problem.

Present partial results, even if your solution is incomplete.

Many sub-problems can be solved independently of previous sub-problems.

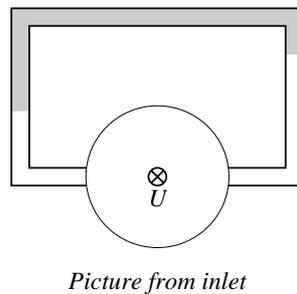
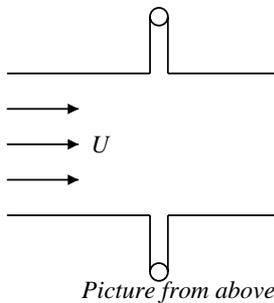
Cartesian coordinates in an inertial frame are used, unless a rotating system is explicitly referred to.

1. Kitchen water velocity (6p)

A friend wants to measure the water velocity in a horizontal pipe, without any disrupting Pitot tube or bottle-neck. My – I mean, *his* – idea is to measure pressure differences due to the Coriolis force.

In the following, assume a water velocity $U \sim 1$ m/s and a pipe diameter ~ 1 dm.

a) [1p] Use order-of-magnitude estimates to discuss if you think Coriolis effects will be measurable in Lund.



The brilliant(?) idea is to attach a small tube as shown in the figure. The tube is in part filled with water with density ρ_0 , and in part (the upper, grey region in the figure) with a fluid with density $\rho_1 < \rho_0$.

b) [3p] A pressure difference between the two sides of the pipe will create a height difference of the surfaces between ρ_0 and ρ_1 , in the two vertical parts of the thin tube. Assuming geostrophic balance, estimate how small the *relative density difference*, $(\rho_0 - \rho_1)/\rho_0$ must be for the Coriolis effect to give a 1 cm height difference.

c) [1p] We should however worry what effects we end up measuring. Suppose there is a small velocity difference u between two sides of the big pipe. For lack of better models, assume constant Bernoulli field H . What u would give roughly the same pressure difference as the one from the Coriolis effect? What is your final verdict about my (...*friend's*) idea?

d) [1p] By the way, what *are* the conditions for constant H ?

2. Flow in a kitchen pipe (6p)

Below are three velocity fields for a flow in the x -direction (along $\hat{\mathbf{e}}_x$), defined in the domain $y^2 + z^2 \leq R^2$ (a cylindrical pipe of radius R around the x axis):

$$\begin{aligned} i) \quad \mathbf{v} &= \exp(-t/\tau)U\hat{\mathbf{e}}_x, \\ ii) \quad \mathbf{v} &= \left(1 - \frac{y^2 + z^2}{R^2}\right)U\hat{\mathbf{e}}_x, \\ iii) \quad \mathbf{v} &= \left(1 - \frac{y^2 + z^2}{R^2}\right)\exp(-x/R)U\hat{\mathbf{e}}_x. \end{aligned}$$

Here, $U > 0$ and $\tau > 0$ are constants of appropriate dimensions.

a) [2p] For each field, determine if it is steady, and if it is irrotational.

b) [2p] Show that only field (ii) can represent a *viscous* flow of an incompressible fluid, inside a pipe at rest with radius R around the x axis.

c) [2p] Assuming constant density ρ_0 , constant viscosity η , and gravity along $\hat{\mathbf{e}}_z$, show that the pressure drop $|\Delta p|$ over a distance Δx in the pipe is

$$|\Delta p| = 4\eta\Delta x \frac{U}{R^2}.$$

3. Tapping kitchen water (6p)



What is the shape of the water jet pouring down from a kitchen tap into a sink?

Assume steady, ideal flow; constant density ρ_0 ; constant gravitational acceleration $-g_0$ along the vertical direction $\hat{\mathbf{e}}_z$; and a cylindrical symmetry of the water jet around the vertical. (So, not a tilted jet of water as in the picture...).

Suitable coordinates are height z above the sink bottom, and horizontal distance r from the jet center. Let the horizontal radius of the jet at height z be $a(z)$. As seen in the picture $|a'(z)| \ll 1$, so we assume $|v_r| \ll |v_z|$, where the velocity field is $\mathbf{v} = v_z(z, r)\hat{\mathbf{e}}_z + v_r(z, r)\hat{\mathbf{e}}_r$.

a) [3p] We may also assume that the pressure is constant in the entire water jet, and that v_z is independent of r . Under all assumptions, show that $a(z) \propto (h - z)^{-1/4}$, where h is an undetermined constant.

b) [2p] Assume constant air pressure p_0 , and that air does not create any shear forces. Looking at boundary conditions, motivate (without rigorous proofs) the assumptions that

pressure is constant in the entire water jet, and that v_z is independent of r .

c) [1p] The parameter h represents a critical height where there is no energy left for vertical motion. As $v_z \rightarrow 0$, the correct total vertical flow can only be obtained if the area $\pi a^2 \rightarrow \infty$. Without calculations, motivate how you think corrections due to v_r would modify the critical height.

4. Flow in a kitchen sink (6p)



Look at the bottom of the sink. There is a sudden change from thin, radial flow to much thicker, less orderly flow. Watching the flow in real life shows that the “edge” is stationary. Can we understand it?

At radial distance r from the jet, the water depth $d(r)$ and the average radial velocity, $\langle v_r \rangle(r)$, must obey

$$r d \langle v_r \rangle = C, \quad (1)$$

where C is a constant. We do not expect that $d(r)$ or $\langle v_r \rangle(r)$ increase with r . Thus, a reasonable model is that $d(r) \propto r^{-\alpha}$, where $0 \leq \alpha \leq 1$.

a) [2p] In the inner, thin region, consider how much water that in time Δt passes through a vertical area with height $d(r)$ at the circumference of a circle of radius r , and verify eq. (1). (You may of course use eq. (1) below even if you did not verify it.)

b) [2p] Near the walls of the sink, the flow is perturbed in many ways, creating surface gravity waves. The dispersion relation for a wave with wavelength $\lambda = 2\pi/k$ traveling on water of depth d can be assumed to be $\omega^2(k) = g_0 k \tanh(kd)$ (where g_0 is the gravitational acceleration).

How does the the phase velocity (celerity) c depend on wavelength for shallow-water waves and deep-water waves, respectively?

c) [2p] The celerity is defined relative to the *water surface*, which itself moves outwards with a velocity that can be assumed to be proportional to $\langle v_r \rangle$. Waves moving inwards will then have a velocity $A\langle v_r \rangle - c$ relative to the *kitchen sink*. (A is a positive constant.) Show that shallow-water waves will stay still relative to the sink at a critical radius R , which depends on unknown constants but is independent of wavelength. Furthermore, show that for $\alpha < 2/3$, waves at $r > R$ move inwards, towards R .

Comment: So, all waves moving inwards from $r > R$ slow down and pile up to form a “sharp edge” at R , which is what we see!

5. Talking about kitchen water... (6p)

Tapping kitchen water creates a lot of funny sounds, propagating in the water. A lecturer once started to talk unprepared about the speed of sound in water, and made a mess of it. Let's help him out!

Look at small, time-dependent perturbations $\Delta\rho$ and Δp to constant values ρ_0 and p_0 of a steady solution. Thus, the fields are $\rho(\mathbf{r}, t) = \rho_0 + \Delta\rho(\mathbf{r}, t)$ and correspondingly for p . Assuming $|\Delta\rho| \ll \rho_0$ and similarly for p , the linearized differential equations (neglecting gravity) of fluid dynamics become

$$\begin{aligned}\frac{\partial\Delta\rho}{\partial t} + \rho_0\nabla\cdot\mathbf{v} &= 0, \\ \frac{\partial\mathbf{v}}{\partial t} &= -\frac{1}{\rho_0}\nabla(\Delta p).\end{aligned}$$

a) [3p] Show that these equations can be combined to a wave equation for density:

$$\frac{\partial^2\Delta\rho}{\partial t^2} = c_0^2\nabla^2\Delta\rho,$$

and describe how c_0 can be determined from a barotropic relation $p = p(\rho)$, where $p_0 = p(\rho_0)$.

b) [1p] The *bulk modulus* $K(\rho)$ of a fluid measures how much pressure increases as a function of relative volume decrease (or relative density increase). It is defined in terms of density and pressure by

$$K(\rho) = \rho\frac{dp}{d\rho}.$$

For water, a good parametrization of K is

$$K(\rho) = K_0 + \gamma[p(\rho) - p_0]. \quad (2)$$

In this model, find an expression for c_0 . In particular, show that c_0 is independent of the constants p_0 and γ , but depends on the constants K_0 and ρ_0 .

c) [1p] If you failed to solve the above sub-problems, use the information that c_0 depends only on K_0 and ρ_0 , and apply dimensional arguments to make an educated guess of what c_0 should be. If you *did* solve (a) and (b), make the dimensional arguments anyway, to score your exam point.

d) [1p] Given values $K_0 \sim 2 \cdot 10^9$ Pa and $\rho_0 \sim 10^3$ kg/m³, make a rough estimate of the speed of sound in water.