Exam, FYTA14, 2017-08-25, 10.00-15.00

Allowed material: One a4 sheet with notes, writing material.

30 points total, 15 points to pass, 24 points for distinction.

The tasks are *not* sorted in order of difficulty.
Read the text *carefully* before you start to solve a problem.
Present partial results, even if your solution is incomplete.
Many sub-problems can be solved independently of previous sub-problems.
Cartesian coordinates in an inertial frame are used, unless a rotating system is explicitly referred to.

1. Newton's bucket (6p)

An ideal fluid rotates in the gravitational field $\boldsymbol{g} = (0, 0, -g_0)$ with constant angular velocity $\boldsymbol{\Omega}$, so that $\boldsymbol{v} = (-\Omega y, \Omega x, 0)$. We search the surfaces of contant pressure, to find the surface of a rotating bucket of water (at constant air pressure).

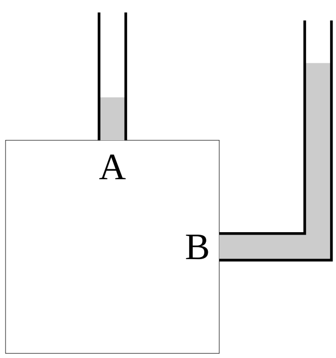
a) [1p] You peek at your class mate's answer during the exam and find that he/she has used Bernoulli's theorem and concluded that the surfaces for constant pressure are

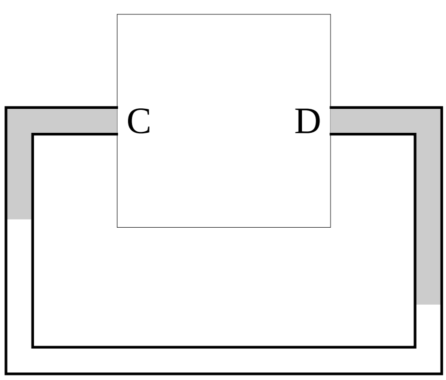
$$z = C - \frac{\Omega^2}{2g_0}(x^2 + y^2).$$

But that would imply that the surface of the water in a rotating bucket would be highest in the centre... what is wrong?

b) [5p] Write Euler's equations in component form, integrate them explicitly to find the pressure $p(x, y, z)$, and determine the correct shape of the water surface.

2. Black box pressure (7p)





Thin tubes are attched to "black boxes" (shown as white boxes...). The grey regions are water in hydrostatic equilibrium. The open tubes are in contact with air at constant pressure. The closed tube is partly filled with a fluid of higher density than water (white region).

a) [3p] Introduce symbols in the figure for variables you need (like height and densities) and derive expressions for the two pressure differences $p_A - p_B$ and $p_C - p_D$. Flat earth gravity is assumed pointing vertically in the figure.

b) [4p] The white boxes could represent setups disrupting an ideal uniform water flow in a pipe. One possible disruption is a pitot tube, creating a stagnation point. Another is a bottle-neck, changing the cross-sectional area for the water flow. Describe how each of these two setups allow the water velocity to be measured through pressure differences.

3. Shear waves (7p)

Consider a simple viscous flow, $\boldsymbol{v} = (v_x(z,t),\, 0,\, 0)$ in a gravitational field $\boldsymbol{g} = (0,\, 0,\, -g_0)$. The velociy v_x is

$$v_x = U_1 \exp(-kz)\cos(\omega t - kz),$$

where U_1, ω and k are constants.

a) [2p] Show that $\boldsymbol{v}$ is consistent with an incompressible fluid with constant dentisty ϱ_0. Furthermore, determine if the flow is steady, and if it is irrotational.

b) [2p] Show that the three components of the Navier–Stokes equation reduce to an equation for hydrostatic equilibrium, and one diffusion equation for v_x.

c) [2p] Suppose the field $\boldsymbol{v}$ is generated by a floor (infinitely extended horizontally) at $z = 0$ moving in the x-direction with velocity $U\cos(\omega t)$ (where U and ω are constants), and that the motion of the fluid vanishes far above ($\boldsymbol{v} \to 0$ as $z \to \infty$). Find expressions for U_1 and k so that this becomes a solution to the Navier–Stokes equation, satisfying the boundary conditions.

d) [1p] For a wave amplitude scaling as $v \propto \exp(-z/\delta)$, the penetration depth δ is a good estimate of how far the wave reaches before being being damped to insignificance. Relate the penetration depth of our shear wave with its wavelength, and discuss why so few underwater animals have developed communication systems based on shear wave generation.

4. Air balloon (3p)

An airborn balloon is drifting in a horizontal wind, which is assumed constant and in geostrophic balance.

a) [1p] How does the air pressure change during the journey?

b) [2p] Using several air balloons, spread out over a large region in different directions, a pressure gradient of about 1 Pa/km is measured, and air density is found fairly constant at $1.2\,\mathrm{kg/m^3}$. Make a rough estimate of how fast the balloons fly.

5. Airplane (7p)

A vessel i is moving with velocity $\boldsymbol{u}_i$ relative to a wind $\boldsymbol{v}$. Thus, the vessel moves with velocity $\boldsymbol{v}_i = \boldsymbol{v} + \boldsymbol{u}_i$ relative to the ground. Both $\boldsymbol{u}_i$ and $\boldsymbol{v}$ are assumed constant and horizontal.

For any scalar field λ, let λ_i be the measurment made from the vessel i. For a stationary field, the time dependence of measurments is then determined by

$$\frac{\mathrm{d}}{\mathrm{d}t}\lambda_i = (\boldsymbol{v}_i \cdot \nabla)\lambda = \boldsymbol{v}_i \cdot (\nabla\lambda).$$

a) [2p] Assume a barotropic atmosphere where there exists a pressure potential $\omega(p)$, satisfying $\omega'(p) = 1/\varrho$. Show that geostrophic balance then gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega_i = 2\boldsymbol{\Omega}_z \cdot (\boldsymbol{v}_i \times \boldsymbol{v}).$$

Hint: The volume product identity $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{a})$ may be useful.

b) [1p] Define the x axis along $\boldsymbol{u}_i$, so that $\boldsymbol{u}_i = (u_x,\ 0,\ 0)$, and show that the total pressure potential change measured as the vessel travels between points A and B is

$$\Delta\omega_i = \omega_B - \omega_A = 2\Omega_z u_x \Delta y_i,$$

where $\Delta y_i = y_B - y_A$ is the vessel drift perpendicular to $\boldsymbol{u}_i$.

c) [1p] Reasonable numerical values are $p \sim 10^5\,\mathrm{Pa}$, $\varrho \sim 1\,\mathrm{kg/m^3}$, $u_x \sim 100\,\mathrm{m/s}$ and $\Delta y_i \sim 10\,\mathrm{km}$. Make a rough numerical estimate of the pressure change $\Delta p_i = p_B - p_A$ in the simple case of constant density.

d) [1p] A polytropic atmosphere is more realistic, where

$$p = p_A \left(\frac{\varrho}{\varrho_A}\right)^{\gamma}.$$

Show that a possible pressure potential is

$$\omega = \frac{\gamma}{\gamma - 1}\frac{p_A}{\varrho_A}\left(\frac{p}{p_A}\right)^{\frac{\gamma-1}{\gamma}}.$$

e) [1p] Assuming $\frac{\gamma-1}{\gamma} \sim 1$ and numerical estimates as before, show that $\Delta\omega_i \ll \omega_A$.

f) [1p] Show that $\Delta p_i \approx 2\varrho_A \Omega_z u_x \Delta y_i$ if $\Delta\omega_i \ll \omega_A$.

Comment: Thus, the constant density approximation for Δp_i may hold also in some reasonable cases of a barotropic atmosphere.