Exam, FYTA14, 2018-06-01, 10.15-15.15

Allowed material: One a4 sheet with notes, writing material.

30 points total, 15 points to pass, 24 points for distinction.

The tasks are *not* sorted in order of difficulty.

Read the text *carefully* before you start to solve a problem.

Present partial results, even if your solution is incomplete.

Many sub-problems can be solved independently of previous sub-problems.

Cartesian coordinates in an inertial frame are used, unless a rotating system is explicitly referred to.

1. Non-dimensional (6p)

Below are three three-dimensional velocity fields given in non-dimensional variables:

$$\begin{aligned} i) \quad & \boldsymbol{v} = (tx, -ty, 0), \\ ii) \quad & \boldsymbol{v} = (xy + xz, -yz, 1 - yz), \\ iii) \quad & \boldsymbol{v} = (\cos x, \sin y, t). \end{aligned} \tag{1}$$

a) [1p] For each one of them, determine if it represents steady flow or not.

b) [2p] For each one of them, determine if it represents irrotational flow or not.

c) [3p] Could any of them be the velocity field for an incompressible ideal flow in the domain $z > 0$, bounded from below by a stationary wall at $z = 0$? If so, compute the corresponding pressure at zero gravity, with the constant non-dimensional density $\rho_0 = 2$.

2. Steadily Down the Drain [6p]

Water flows vertically down a rectangular shaft. The shaft is much wider in one direction, so we approximate it as two parallell planes, separated by a distance $2d$ in the x-direction, and disregard the boundary conditions at the walls far away in the y-direction. Looking for a steady solution, we can then assign $\boldsymbol{v} = (0,\ 0,\ v_z(x))$.

Assuming constant pressure, and gravity $\boldsymbol{g} = (0,\ 0,\ -g_0)$, find the velocity $v_z(x)$ that solves the Navier–Stokes equation and satisfies boundary conditions at the considered walls. Let them be at $x = d$ and $x = -d$, respectively.

3. Half Empty (7p)

A large, open tank with constant cross-section A_0 and height h is filled with water. At time $t = 0$, an outlet with cross-section $A < A_0$ is opened at the bottom, and water exits with a velocity v, while the surface in the tank sinks with a velocity v_0.

a) [1p] For steady, ideal flow and constant gravity, find a relation between v, v_0 and z, where z is the height difference between the outlet and the water surface.

b) [1p] Assuming incompressible flow, find a relation between v and v_0, independent of z.

c) [1p] Combining the above assumptions, express the velocity through the outlet as a function of z.

d) [3p] Allow the water surface height $z(t)$ to vary with time, $\frac{\mathrm{d}z}{\mathrm{d}t} = -v_0$, $z(0) = h$, despite the assumption of steady flow (a "quasi-stationary" approach). Find $z(t)$. What is $T_{1/2}/T$, where $T_{1/2}$ is the time required for the tank to empty halfway and T the time required for the tank to empty completely?

e) [1p] Hopefully, your results in (c) and (d) are un-physical in the limit $A \to A_0$, representing a situation where almost the whole bottom of the tank is opened as an outlet. What assumption(s) above would be un-reasonable for large A? (It is possible to answer this question without any answers in (c) and (d) to refer to.)

4. Tilted (6p)

Assume a steady, uniform current of water flowing horizontally with speed U, so that the velocity field is $\boldsymbol{v} = U\widehat{\boldsymbol{e}}_x$, where the x-direction is defined along the current.

Due to the Coriolis force, the water surface will not be quite horizontal, despite a vertical gravitational field $\boldsymbol{g} = -g_0\widehat{\boldsymbol{e}}_z$.

a) [1p] Where will the slope be most noticeable (assuming no disturbances such as winds, or boats, or waves, or...), in Öresund between Sweden and Denmark, or in the Suez canal in Egypt? Motivate your answer.

b) [4p] Assuming geostrophic balance, incompressible water and constant air pressure, find an expression for the water surface. You will need to introduce additional symbols for various physical quantitites.

c) [1p] A reasonable water velocity is $U \sim 1\,\mathrm{m/s}$, and a uniform flow can hardly be a good approximation at length scales above $L \sim 10^4\,\mathrm{m}$. Near Lund, the local angular velocity is $\Omega_z \sim 0.6 \cdot 10^{-4}\,\mathrm{s}^{-1}$. Estimate the Rossby number for the system. Does it motivate consideration of the Coriolis force? Is the Rossby number relevant for the proposed flow?

5. Shallow (5p)
Consider ideal water flow in constant gravity $\boldsymbol{g} = -g_0\widehat{\boldsymbol{e}}_z$. Assume constant water density ρ_0.

a) [1p] Show that if the advective term can be neglected, Euler's equation becomes

$$\frac{\partial}{\partial t}\boldsymbol{v}(\boldsymbol{r},t) = -\nabla\left[\Phi(z) + \frac{1}{\rho_0}p(\boldsymbol{r},t)\right],$$

where p is the pressure. Give an expression for the function $\Phi(z)$.

b) [2p] Show that $\Phi^* = \Phi + p/\rho_0$ satisfies the Laplace equation $\nabla^2\Phi^* = 0$.

The water surface height h is described by a harmonic wave of amplitude a moving in the x direction, $h = a\cos(kx - \omega t)$, where $k > 0$ and $\omega > 0$ are constants. We therefore expect

$$\Phi^*(x,z,t) = f(z)\cos(kx - \omega t + \theta) + C,$$

where θ and C are constants and $f(z)$ is an unknown function of z.

c) [2p] Find θ and $f(z)$ so that Φ^* solves the Laplace equation and the derivative $f'(z)$ satisfies the boundary conditions $f'(-d) = 0$ and $\frac{\partial z}{\partial \Phi}^*(x,z,t)\big|_{z=0} = kA\sinh(kd)\cos(kx - \omega t)$, where A is an undetermined constant. Consider the range $-d \leq z \leq 0$ and show that the shallow water limit, $kd \ll 1$, makes $f'(z)$ approximately a linear function of z. (Note: you do not need to motivate the given boundary conditions. That's for another exam.)