Exam, FYTA14, 2018-08-31, 8.00-13.00

Allowed material: One a4 sheet with notes, writing material.

30 points total, 15 points to pass, 24 points for distinction.

The tasks are *not* sorted in order of difficulty.
Read the text *carefully* before you start to solve a problem.
Present partial results, even if your solution is incomplete.
Many sub-problems can be solved independently of previous sub-problems.
Cartesian coordinates in an inertial frame are used, unless a rotating system is explicitly referred to.

1. Flows on floor (8p)
Below are three three-dimensional velocity fields given in non-dimensional variables:

$$\begin{aligned} i) \quad & (z + 2xy,\, x^2 - y^2,\, x), \\ ii) \quad & (x,\, -y,\, 0), \\ iii) \quad & (xz,\, -yz,\, 0). \end{aligned}$$

a) [1p] Determine if the flows are incompressible or not.

b) [4p] Show that only one of them can be the velocity field for an irrotational ideal flow in the domain $z > 0$, bounded from below by a stationary wall at $z = 0$. For that flow, compute the corresponding pressure at zero gravity, with the constant non-dimensional density $\rho_0 = 2$ and condition $p = 2$ at $x = y = z = 0$.

c) [3p] Show that only one of them can be the velocity field for a viscous flow in the domain $z > 0$, bounded from below by a stationary wall at $z = 0$. For that flow, use the stress tensor $\boldsymbol{\sigma}$ to compute the drag force per area on the floor, $\boldsymbol{S} = (\sigma_{xz},\, \sigma_{yz},\, 0)$, with constant non-dimensional viscosity $\eta = \frac{1}{2}$.

2. Geostrophic wind (8p)

For Geostrophic Balance, the wind obeys

$$0 = \boldsymbol{g} - \frac{1}{\rho}\nabla p - 2\boldsymbol{\Omega}_z \times \boldsymbol{v}. \tag{1}$$

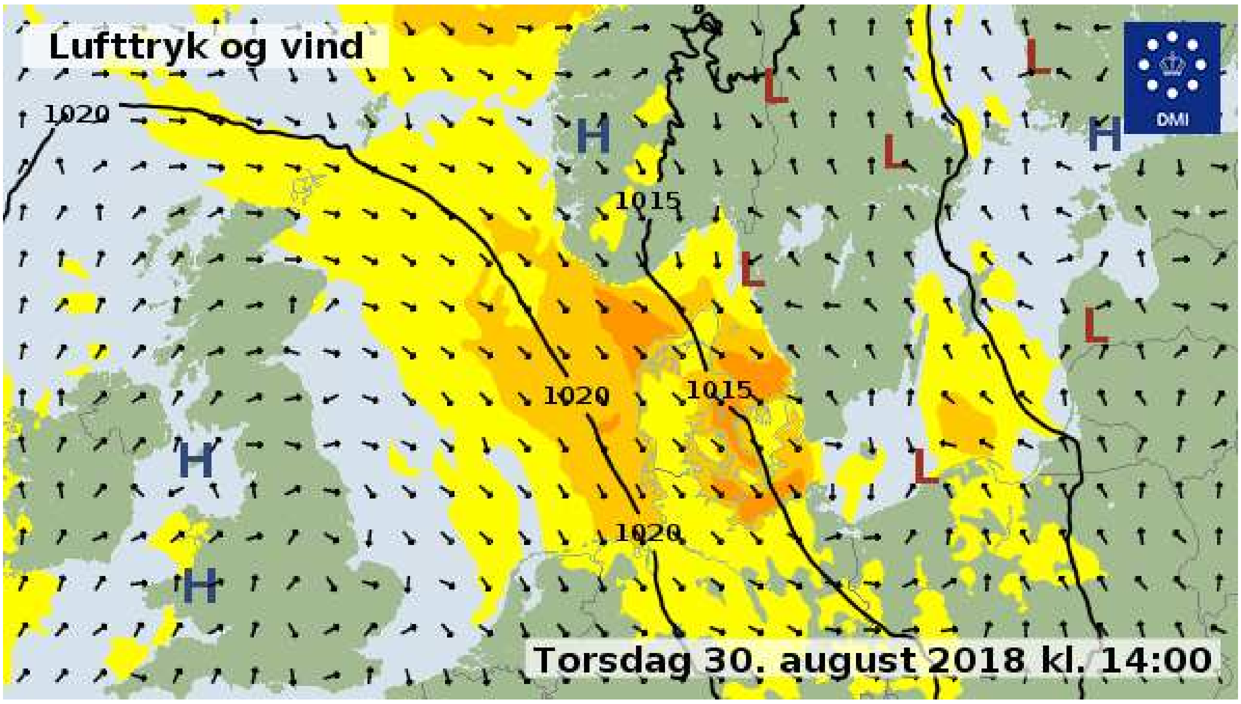
Here, the "local angular velocity" $\boldsymbol{\Omega}_z = \Omega_z \widehat{\boldsymbol{e}}_z$ is the vertical component of the earths angular velocity vector.

a) [1p] As a preparation for physics applications, first show the mathematical relation

$$\widehat{\boldsymbol{e}}_z \times (\widehat{\boldsymbol{e}}_z \times \boldsymbol{u}) = -(u_x,\, u_y,\, 0).$$

b) [2p] Apply the cross product, $\widehat{\boldsymbol{e}}_z \times$, to eq. (1), and find an expresion for the horizontal wind $\boldsymbol{v}_h = (v_x,\, v_y,\, 0)$.

c) [1p] For a vertical gravitational field, $\boldsymbol{g} = -g_0 \widehat{\boldsymbol{e}}_z$, find a simple relation between $\boldsymbol{v}_h$ and ∇p.



d) [1p] According to this DMI forecast from 2018-08-27, there was yesterday a pressure drop $\Delta p \sim 5\,\text{hPa}$ over a distance $\Delta L \sim 200\,\text{km}$ across Denmark. Assuming $\rho \sim 1\,\text{kg/m}^3$ and $\Omega_z \sim 0.5 \cdot 10^{-4}\,\text{s}^{-1}$, estimate a typical magnitude of the gestrophic wind, $|\boldsymbol{v}_h|$, over Denmark. You may assume magnitude of the horizontal part of ∇p to be $(\Delta p)/(\Delta L)$.

e) [3p] The DMI homepage informs that the colours over Denmark represent a typical wind speed $\sim 10\,\text{m/s}$, which is lower than the geostrophic solution in (d). Also, the arrows showing the wind direction are pointing a bit towards lower pressure in some regions (over Norway and northern Germany, for example), which is in disagreement with the geostrophic wind direction $\boldsymbol{v}_h$ in (c). However, the DMI map shows winds near the earth surface, where

Geostrophic Balance does not hold perfectly. Discuss if you think the DMI map reveals expected "near ground" corrections, both in wind strength and wind direction.

Hint: Note that the qualitative corrections are stated here in sub-problem (e). You can discuss the differences even if you did not find answers in (c) and (d). If you do not see the described differences to your answers in (c) and (d), do have a look at those sub-problems again and see if you find any simple error!

3. Sound solution (7p)
Consider an ideal, compressible gas close to an equilibrium where the velocity field is $\boldsymbol{v}_0 = 0$ and density ρ_0 and pressure p_0 are constants. Let $\boldsymbol{v}$, $\Delta\rho$ and Δp denote the deviations from the equilibrium values, so that $\rho = \rho_0 + \Delta\rho$, etc.

a) [1p] Assume a barotropic relation $p = p(\rho)$. Use Taylor expansion to show that

$$\Delta p \approx c_0^2 \Delta\rho$$

and define the velocity constant c_0.

b) [3p] When all time dependent corrections are small, $\Delta\rho$ is governed by a wave equation,

$$\frac{\partial^2}{\partial t^2}(\Delta\rho) = c_0^2 \nabla^2 (\Delta\rho). \qquad (2)$$

Consider a one-dimensional case, $\Delta\rho = \Delta\rho(x,t)$. Assign separation of variables, $\Delta\rho(x,t) = T(t)f(x)$, and show that in order to solve eq. (2), the introduced single-variable functions must satisfy

$$\begin{aligned} T''(t) &= -\omega^2 T(t), \\ f''(x) &= -k^2 f(x), \end{aligned}$$

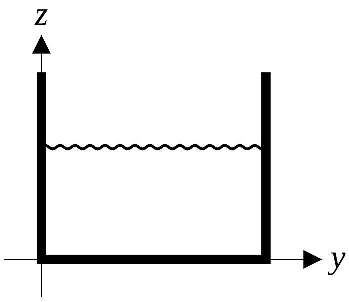
and find a relation between the undetermined constants ω^2 and k^2. (Comment: for oscillating solutions, ω is real and by convention positive.)

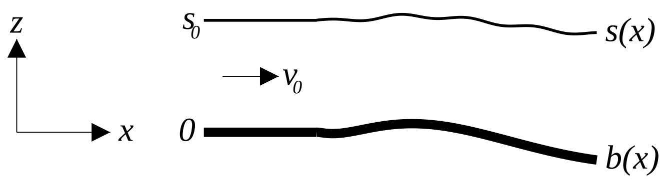
c) [1p] With both ω and k real and positive, write down a general solution for $\Delta\rho$, involving four undetermined constants.

d) [2p] Find the $\Delta\rho(x,t)$ that satisfies the boundary conditions $\Delta\rho = \rho_1 \cos(\omega t)$ and $\frac{\partial}{\partial x}\Delta\rho = k\rho_1 \sin(\omega t)$, where ρ_1 is the constant amplitude of the density fluctuations.

4. Canal Surface (7p)
A canal has vertical walls and constant width. It is directed in the x-direction and the vertical is in the z direction. The gravitational field is thus $\mathbf{g} = (0,\ 0,\ -g_0)$ with $g_0 \approx 10\,\mathrm{m/s^2}$.

Assume that water in the canal obeys steady, ideal, incompressible flow, and that the dynamics is independent of the position y across the canal.





In an upstream region, the bottom is flat at $z = 0$ and the surface is flat at $z = s_0$. The velocity in this region is uniform, $\mathbf{v} = (v_0, 0, 0)$. A bit downstream, however, the bottom follows a curve $b(x)$, and the surface is then described by a function $s(x)$. For simplicity, assume that v_x in this region is independent of z, so that

$$\boldsymbol{v} = \big(v_x(x),\ 0,\ v_z(x,z)\big).$$

a) [1p] Use incompressibility to motivate that the expression $v_x(x)[s(x)-b(x)]$ is independent of x.

b) [1p] Use boundary conditions to show that the flow at the surface satisfies $v_z(x,s) = s'(x)v_x(x)$.

c) [3p] Assuming constant air pressure, show that the functions $s(x)$ and $b(x)$ must satisfy

$$(1+s'^2)C\frac{s_0^2}{(s-b)^2} + s = C + s_0. \qquad (3)$$

Express C in the constants provided in the problem description.

Hint: Follow a stream line at the surface. Make use of the stated relations in previous subproblems, even if you failed to verify them.

d) [2p] Suppose the surface has negligeble curvature, i.e., $s''(x) \approx 0$. With $s_0 \approx 1\,\mathrm{m}$, $v_0 \approx 1\,\mathrm{m/s}$, and $s \approx 1\,\mathrm{m}$, will the surface and bottom have opposite slopes or not? Opposite slopes means that the surface gets lower ($\mathrm{d}s < 0$) over a bump at the bottom ($\mathrm{d}b > 0$). An unmotivated guess will not reward any points.

Hint: The relation in eq. (3) can be used to write b as a function of s and s'. With $s'' \approx 0$, you can treat s' as a constant, and calculate $\mathrm{d}b/\mathrm{d}s$ to find the relative signs of the slopes. If you lack an expression for C, you can solve most of the problem and state a partial result as a condition on C.