Exam, FYTA14, 2019-06-07, 10.00-15.00

Allowed material: One a4 sheet with notes, writing material.

30 points total, 15 points to pass, 24 points for distinction.

The tasks are *not* sorted in order of difficulty.

Read the text *carefully* before you start to solve a problem.

Present partial results, even if your solution is incomplete.

Many sub-problems can be solved independently of previous sub-problems.

Cartesian coordinates in an inertial frame are used, unless a rotating system is explicitly referred to.

1. Pressure in a Tornado (9p)

The "Rankine vortex" can be used as a simple model for tornadoes. It assumes the flow to be

$$\boldsymbol{v} = \begin{cases} \boldsymbol{v}^{(core)} &= \omega_0(-y,\, x,\, 0), \;\; r \leq R \\ \boldsymbol{v}^{(out)} &= \omega_0 \frac{R^2}{r^2}(-y,\, x,\, 0).\; r > R \end{cases} . \tag{1}$$

Here, ω_0 and R are constants, and r is the horizontal distance to the tornado center:

$$r = \sqrt{x^2 + y^2}.$$

Thus, it assumes the velocity to be independent of height z.

We will assume that $\boldsymbol{v}$ describes ideal flow in a vertical gravitational field $\boldsymbol{g} = (0,\, 0,\, g_z)$. We also assume constant density ρ_0, for simplicity.

a) [1p] Show that the z-component of Euler's equation reduces to an equation for hydrostatic equilibrium.

b) [3p] For both the core region ($r \leq R$) and the outer region ($r > R$), determine if the flows are steady, incompressible and/or irrotational, respectively. Be careful to handle product rules and chain rules correctly with the factor $\frac{1}{r^2} = \frac{1}{x^2+y^2}$.

c) [1p] Your results in (b) should allow you to conclude that the Bernoulli field H is constant for one of the regions, but not the other. Determine which one. Alternatively, if your results in (b) do not lead to the correct conclusion, discuss what would need to change.

d) [3p] Calculate the pressure in each of the regions. The answers should contain one undetermined constant each.

e) [1p] Assuming continuous pressure at $r = R$, show the difference $p(r = 0) - p(r = \infty) = -\rho_0\omega_0^2 R^2$.

Comment: This illustrates the possibility of very low pressure in the centre of a tornado!

2. The Gulf Stream (6p)

The Gulf Stream runs counter-clockwise in the northern Atlantic Ocean (as a very simplified description). According to Wikipedia it is typically 100km wide and flows with a velocity of about 1 m/s. We assume that the current can be described by *geostrophic balance*, and that the water surface lies at constant pressure p_0.

a) [2p] Describe why there will be a height difference between the inner and outer edges of the current, and determine which edge that has the highest surface. Furthermore, assuming (boldly) that the width and speed of the flow is constant, where on earth would the height difference be maximized?

b) [2p] Deeper down, the Gulf stream meets other water layers with different motion. At the lower regions of the Gulf stream there is therefore an Ekman layer where the flow disagrees with geostrophic balance. Qualitatively, what is the direction of the flow in the Ekman layer, and how does that affect the height difference between the outer and inner water surfaces?

c) [1p] If the stream instead had run clock-wise, with preserved numerical values, would the height difference change in size or direction? Would the effects of the Ekman layer change? For the "yes" answers: what is the change?

d) [1p] *A side-remark stated as a sub-problem*: When the counter-clockwise Gulf stream turns left, you may expect a pressure gradient to drive the change of velocity direction. With a typical speed U and radius R of the turn, the acceleration "left-wards" is U^2/R. With reasonable assumptions on R, show that this acceleration is negligible compared to other effects in geostrophic balance. Thus, this acceleration need not be considered in the rest of the problem!

3. Flow on a Plane (7p)

Consider steady water flow in the x-direction on a plane perpendicular to the z-direction, such that $\mathbf{v} = (v_x(z), 0, 0)$. The water surface is at constant height above the plane. The plane is lying still. Assume constant water density ρ_0 and a constant pressure p_0 at the surface.

Let the plane (and the coordinate system) be tilted with respect to the earth vertical, so that the gravitational acceleration is $\mathbf{g} = (g_x, 0, g_z)$, where g_x and g_z are constants.

a) [1p] Verify that the proposed velocity field is consistent with a constant density.

b) [2p] Show that the pressure in the fluid is independent of x. *Hint:* Use the z-component of the Navier–Stokes equations and the pressure boundary condition at the surface.

c) [1p] Knowing that the pressure is independent of x (even if you have not proven it), find an expression for $v_x''(z)$.

d) [3p] Find an expression for $v_x(z)$ and determine all integration constants with the help of two boundary conditions:
1) The velocity at the plane
2) The shear forces at the surface. Assume that air imposes negligible shear forces on the water, so that the stress tensor component σ_{xz} is 0 at the water surface.

4. Sound Approximations (8p)

Consider an ideal, compressible gas with small, time-dependent corrections to a steady solution, so that $\rho = \rho_0(\boldsymbol{r}) + \varepsilon\rho_1(\boldsymbol{r}, t)$, $p = p_0(\boldsymbol{r}) + \varepsilon p_1(\boldsymbol{r}, t)$ and $\boldsymbol{v} = \boldsymbol{v}_0(\boldsymbol{r}) + \varepsilon\boldsymbol{v}_1(\boldsymbol{r}, t)$. Here, a dimensionless number $\varepsilon \ll 1$ has been introduced to emphasize that the time-dependent corrections are small, *e.g.*, $|\varepsilon\rho_1| \ll \rho_0$. To simplify, we assume $\boldsymbol{v}_0 = 0$.

During the course, we encountered an even simpler example, with zero gravity and constant p_0, ρ_0. Here, we will allow for space dependence $p_0(\boldsymbol{r})$ and $\rho_0(\boldsymbol{r})$ and introduce a constant gravity field $\boldsymbol{g}$.

We assume that there exists a barotropic relation $p = f(\rho)$, and define the static fields so that $p_0(\boldsymbol{r}) = f(\rho_0(\boldsymbol{r}))$.

a) [1p] Use Taylor expansion of $f(\rho)$ to show that

$$\rho_1 = \frac{1}{c_0^2} p_1$$

and define the velocity c_0.

b) [3p] Taylor expand the equation of continuity and Euler's equations in ε, and show that the $\mathcal{O}(1)$ terms become one trivial $0 = 0$ and

$$0 = \boldsymbol{g} - \frac{1}{\rho_0}\nabla p_0, \qquad (2)$$

while the $\mathcal{O}(\varepsilon)$ terms result in the equations

$$\begin{aligned} \frac{\partial}{\partial t}\rho_1 + \nabla\cdot(\rho_0\boldsymbol{v}_1) &= 0, & (3) \\ \frac{\partial}{\partial t}\boldsymbol{v}_1 &= \frac{\rho_1}{\rho_0^2}\nabla p_0 - \frac{1}{\rho_0}\nabla p_1. & (4) \end{aligned}$$

c) [3p] Study $\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}p_1 = \frac{\partial^2}{\partial t^2}\rho_1$ and show that the three equations stated in (b) combine to

$$\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}p_1 = \nabla^2 p_1 - (\boldsymbol{g}\cdot\nabla)\frac{p_1}{c_0^2}. \qquad (5)$$

d) [1p] Without the $\boldsymbol{g}$-dependent term, the result in (c) is just a wave equation, and we expect solutions of the form $p_1 \propto \sin(kx - \omega t)$ where $\omega = c_0 k$. Thus, the relative correction due to $\boldsymbol{g}$ can be estimated by $\frac{|\boldsymbol{g}\cdot\nabla p_1|}{|\frac{\partial^2}{\partial t^2}p_1|} \sim \frac{gk}{\omega^2}$. The lower audible angular frequencies ω for a human ear are about $100\,\mathrm{s}^{-1}$. Use your best estimates (they do not have to be good) about g and c_0 to discuss if you think the gravity-dependent term needs to be considered in everyday applications about audible sound.