Written exam, FYTA14, 2019-08-29, 08.00–13.00

Allowed material: One a4 sheet with notes, writing material.

30 points total, 15 points to pass, 24 points for distinction.

The tasks are *not* sorted in order of difficulty.

Read the text *carefully* before you start to solve a problem.

Present partial results, even if your solution is incomplete.

Many sub-problems can be solved independently of previous sub-problems.

Cartesian coordinates in an inertial frame are used, unless a rotating system is explicitly referred to.

1. Surface Gravity Waves (5p)

a) [1p] Make a sketch of a surface gravity wave and the relevant physical quantities that are important for its propagation.

b) [2p] Give a qualitive description of what drives the propagation of a surface gravity wave.

c) [2p] Describe the difference between the properties of a shallow-water wave and a deep-water wave.

2. Geostrophic stream function (6p)

For a two-dimensional flow, $\boldsymbol{v} = (v_x,\, v_y,\, 0)$ it is sometimes possible to define a *stream function* ψ that satsifies $v_x = \frac{\partial}{\partial y}\psi$ and $v_y = -\frac{\partial}{\partial x}\psi$.

a) [1p] Show that if a stream function ψ exists, then the 2-dimensional velocity field $\boldsymbol{v}$ must represent incompressible flow.

b) [1p] The change of any field λ along a stream line is determined by the co-moving derivative $D\lambda/Dt$. Show that for steady flow, the stream function ψ is constant along stream lines.

c) [3p] Consider a horizontal wind in geostrophic balance. Assume there is a barotropic relation between pressure p and density ρ such that a pressure potential ω solving $\omega'(p) = 1/\rho$ can be defined. Show that ω multiplied by an appropriate constant is a stream function.

d) [1p] Comment on how the claims in (b) and (c) are related to well-known relations between pressure and winds in geostrophic balance! Note that you can discuss the claims even if you have not verified them.

3. Ideal play (8p)

A child pours water into a chute (ränna) and watches it flow down to the grass. The chute is inclined by an angle α. It has flat bottom and vertical walls at a constant distance B from each other.

In the following we use a coordinate frame where the x-axis is pointing in the direction of the flow, and the z-axis is normal to the bottom of the chute. Thus, gravity is **not** pointing in the z-direction, but is given by $\mathbf{g} = (g_0 \sin\alpha,\ 0,\ -g_0 \cos\alpha) = (g_x,\ 0,\ g_z)$.

The precise child adds water so that the water depth at the top (measured in the z-direction) is constantly h_0 and the flow velocity is constantly v_0. The velocity v_0 is independent of z.

Assume ideal, steady flow. Assume also constant water density and that the water velocity in the z-direction can be neglected: $v_z \approx 0$.

Consider a position x along the chute where the water has depth h in the z-direction and velocity v_x in the x-direction.

a) [1p] Determine how pressure p in the water depends on z and h. Assume constant air pressure p_0.

b) [3p] Every stream line starts somewhere at the top of the chute. Show that the Bernoulli field H depends on v_x, x and h, but not on z. (You will be needing the expression for pressure from (a).)

c) [1p] Find an expression relating $v_x(x)$ and $h(x)$, that depends on the constants v_0 and h_0.

d) [2p] With values $v_0 \approx 1\,\mathrm{m/s}$, $h_0 \approx 1\,\mathrm{cm}$ and $\sin\alpha \approx 0.1$, approximately how far should the chute be for the water depth at the end to be half the depth at the top? (With so many approximate values, do not hesitate to make additional rough approximations, when needed, to find a value without a calculator.)

e) [1p] The largest deviation from our approximation $v_z \approx 0$ will appear at the water surface, where we have $v_z(h) = v_x h'(x)$. Using the values of v_0, h_0 and $\sin\alpha$ from (d), show that $|h'(x)| \ll 1$. (Comment: In this way we can in hindsight justify the approximation $v_z \approx 0$.)

4. Frictional play (8p)

Consider the same game and the same assumptions as in the last problem ("ideal play"), but with one important difference: do not assume ideal flow, but include viscosity and assume laminar flow.

Use the same coordinate frame as in the previous problem.

a) [2p] The velocity v_x will depend on z. A bit downstream, v_x will instead become independent of x. The water depth h thus becomes constant. Motivate these claims about v_x and h!

b) [3p] Let the flow be steady. For simplicity assume that friction against the walls can be neglected (but consider friction against the bottom). Show that the velocity v_x as a function of the distance z to the bottom of the chute obeys

$$v_x''(z) = C_0. \tag{1}$$

Express C_0 in constants given in the problem.

c) [2p] Find the most general form for $v_x(z)$ that solves eq. (1) and satisfies the boundary condition at the bottom of the chute. (You may use eq. (1) even if you failed to show it. If so, consider C_0 a "known" constant, that cannot be adjusted to any boundary conditions.)

d) [1p] In task 3 and 4 we have treated the same problem with two different models. The kinematic viscosity of water is $\nu \sim 10^{-6}\,\mathrm{m^2/s}$. With a typical water depth $h \sim 1\,\mathrm{cm}$ and velocity at the surface $v_x(h) \sim 1\,\mathrm{m/s}$, which of the models is most relevant? Or is it neither? Motivate your opinion.

5. Generalized Bernoulli hotness (3p)

When there is a barotropic relation such that density ρ can be written as a function of pressure p, it is possible to define a pressure potential ω, solving $\omega'(p) = 1/\rho$.

a) [2p] For an ideal, polytropic gas, temperature T, density and pressure are related by $p = \rho R T$ and $p \propto \rho^\gamma$. Here, R and γ are constants. Show that T is a constant times ω.

b) [1p] Consider an object in a uniform horizontal wind. Far away, the wind is $\boldsymbol{U}$, the pressure is p_0 and the temperature is T_0. One horizontal stream line ends in a stagnation point on the object surface. Find an expression for the difference between T_0 and the temperature in the stagnation point. You may use the claim in (a) even if you failed to determine the constant.