# Comments on the turbulence lecture

First, we noted that even if we can find solutions with laminar flow, they need not be *stable*. A simple analogy is a ball at rest in a landscape. If the ball is at the bottom of a valley, the solution is stable, but when the ball is at rest on the top of a hill, the solution is unstable: any deviation from the solution will bring the ball further and further away.

It is the non-linear advective term in Navier–Stokes equations that give rise to instabilities. In simple words, the advective term is part of the comoving derivative and represents the fact that forces act on already moving mass. This means that the effect of forces trying to correct deviations from a laminar solution may end up "in the wrong place". The result is turbulence.

The lecture was build around the figure in the margin of page 565, which shows that turbulent flow can have a thinner boundary layer than laminar flow.

**Eddies**
First, we discussed *eddies*, some "chunks" of fluid with laminar flow within, but with drastically different flow from neighbouring eddies.

An eddie of size $\lambda$ and typical flow velocity $u_\lambda$ will likely have a typical internal velocity difference also of the order $u_\lambda$. After a time $\tau_\lambda = \lambda/u_\lambda$, different parts in the same eddie have distanced themselves by $\lambda$. As this is the size of the entire eddie, the parts can no longer belong to the same eddie, and it has by then split into smaller eddies. This is a simple argument for a cascade model, where big eddies split into smaller and yet smaller eddies.

For really small eddies of size $\lambda_d$, the time $\tau_d$ it takes for them to split is comparable to the time $\tau_\nu \equiv \lambda_d^2/\nu$ it takes for normal viscosity to smear out any velcty differences (by diffusion as from eq. (15.5)). Those eddies never really split, but transfer their kinetic energy into heat. In the cascade model, it is only these small eddies that are responsible for dissipation (energy loss to heat). Larger eddies transfer their kinetic energy to smaller ones, but the energy is still kinetic. That is a nice and simple model. The "specific dissipation rate" $\varepsilon$ (kinetic energy lost to heat per time and mass) is then determined by the smallest eddies alone.

Kolmogorov's scaling law is based on the assumption that the dynamics of larger eddies, in particular the "specific turnover rate" (energy lost to smaller eddies per time and mass) is indepondet of the system scale $L$ and only dependent on eddie size $\lambda$ and the observable $\varepsilon$. The kinetic energy per mass, $\frac{1}{2}u_\lambda^2$ is lost to smaller eddies in time $\tau_\lambda$, so we get $u_\lambda^2/\tau_\lambda \sim \varepsilon$. Furthermore, the mass $\mathrm{d}M$ carried by eddies of sizes between $\lambda$ and $\lambda + \mathrm{d}\lambda$ should be proportial to $\mathrm{d}\lambda$ and to the total mass $M$. To get dimensions right, we find $\mathrm{d}M \propto (M/\lambda)\mathrm{d}\lambda$. This model contains many vague assumptions, but the main message is that, put together, the assumptions predict an energy spectrum $\mathrm{d}E/\mathrm{d}\lambda$ which can be confronted with data. (Just as with electromagnetism, we can construct measuring devices that mainly absorb energy from Fourier components for wavelengths $\lambda$.)

Turbulent flow will diffuse velocity differences, typically more efficiently than the molecular

processes behind the kinematic viscosity $\nu$. Their effective kinematic viscosity (diffusion constant) should scale as $\lambda u_\lambda$ (if nothing else so for dimensional reasons). Looking at the scaling from the cascade model, one finds that the largest eddies, of size $\lambda \sim L$, dominate the diffusion, giving rise to an effective kinematic viscosity $\nu_{\text{turb}} \sim u_L L \sim UL$. The transition between turbulence and laminar flow should determined by the ratio of kinematic viscosities (so that the dominant term specifies the flow). In other words, the flow is characterized by $\nu_{\text{turb}}/\nu \sim UL/\nu$, which indeed is the Reynolds number!

(As a side note, in physics when you use simple dimensional arguments for critical quantities, such as $UL/\nu$, the unknown constant to be used in front is often of order 1. Our current case is an exception: even if the turbulent kinemitc viscosity scales as $UL$, the factor in front is apparently very small, so that $\nu_{\text{turv}} \sim \nu$ implies $UL \sim 10^4 \nu$, which by experience is the transition between turbulence and laminar flow.)

**Additional reading**:
• pages 561-568, but skip "Specific rate of dissipation". It is enough to know that $\varepsilon$ is the energy loss per time and mass. The section involves how to calculate $\varepsilon$ in pipe flow, but we do not need that for the rest of the chapter.
• "Eddie viscosity" on page 569.

**Drag**
Air resistance is typically modeled as a force proportional to the wind velocity $U$, or proportional to $U^2$. Which model to use depends on the size and velocity of the object experiencing the wind.

For a still object in a wind obeying laminar flow, the air resistance must be proportional to $\eta$ and the length $L$ of the object (which determines how long wind is slowed down by the no-slip condition at the object boundary). The remaining quantity it can depend on is the wind speed $U$, and to get dimensions right, the drag force becomes $F \propto \eta L U$, so it is proportional to $U$.

For turbulent flow, a simple model is to say that all the wind hitting the object with speed $U$ makes its turbulent passing somehow and ends up in rest behind the object. In time $\Delta t$, the mass $\rho L^2 U \Delta t$ hits (and passes) the object, and loses momentum $\Delta p = (\rho L^2 U \Delta t)U$. The momentum loss per time gives the force $F = \Delta p/\Delta t \propto \rho L^2 U^2$.

Which process that dominates is determined by the ratio $\rho L^2 U^2/(\eta U L) = UL/\nu$. Not surprisingly, we found the Reynolds number again!

Very surprisingly, on the other hand, is that the drag does not grow monotonically with $U$. In the transition from drag linear in $U$ to drag linear in $U^2$, the force drops. This is known as the "drag crisis" and golfers exploit it every day! An ancient, smooth golf ball flies with a Reynolds number slightly below turbulent regimes. The dimples in modern balls help create turbulence, and the ball flies longer!

The reason for the drag crisis is that turbulent flow actually results in thinner boundary layers which leave a smaller wake behind the flying object. One could naively expect that larger $\nu$ would give thicker boundary layers, but then we neglect the fact that $\nu_{\text{turb}}$ need not

be constant. In fact, a distance $\delta$ from a wall, we must expect the largest eddies that can contribute to diffusion have $\lambda \sim \delta$. Since turbulent viscosity grows with eddie size, this gives smaller effective $\nu_{\text{turb}}$ close to the boundary.

**Additional reading:**
• pages 294-296. Do not focus on math, but on figures and reasoning.