Problems Chapter 4.1-4.6, QFT 2013

1. Show that the time evolution operator $U(t, t_0) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)}$ is a solution to the Schrödinger type differential equation

$$i\frac{\partial}{\partial t}U(t, t_0) = H_1(t)U(t, t_0)$$

where $H_1(t) = e^{iH_0(t-t_0)}H_{int}e^{-iH_0(t-t_0)}$ and $H_{int} = H - H_0$. (Note that both $H_0$ and $H_{int}$ are time-independent)

2. Show that

$$T\{\phi(x)\phi(y)\} = N\{\phi(x)\phi(y) + D_F(x - y)\}$$

where $D_F(x - y) = \phi(x)\phi(y)$

3. Show that

$$|\vec{p}_A|E_B + |\vec{p}_B|E_A = \sqrt{(p_A \cdot p_B)^2 - m_A^2 m_B^2}$$

This shows that $E_AE_B|v_A - v_B|$ is Lorentz invariant if $v_A = \vec{p}_A/E_A$, $v_B = \vec{p}_B/E_B$ and the two momenta are in opposite directions along the same axis. If $v_A = p_A^x/E_A$, $v_B = p_B^x/E_B$ such that the two momenta are not along the same axis, then it is Lorentz invariant under longitudinal boosts.

4. Show that in the centre of mass frame the two-particle phase space can be written as

$$\int d\Pi_2 = \int d\Omega_1 \frac{1}{16\pi^2} \frac{1}{E_{cm}} |\vec{p}_1|$$

where $E_{cm} = E_A + E_B$ is the total energy of the colliding particles.

5. Show that

$$\phi_i^+(x)|\vec{p}\rangle_0 = e^{-ip^x x}$$

where $\phi_i^+(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} a_\vec{p} e^{-ip^x x}$.

6. Show that

$$\langle 0| a_{\vec{p}_A}^1 a_{\vec{p}_B}^1 a_{\vec{p}_B}^1 a_{\vec{p}_A}^1 |0\rangle = (2\pi)^6 \left[ \delta^{(3)}(\vec{p}_A - \vec{p}_1) \delta^{(3)}(\vec{p}_B - \vec{p}_2) + (1 \leftrightarrow 2) \right]$$

7. Consider the following Lagrangian for two real scalar fields, $\Phi$ and $\phi$

$$\mathcal{L} = \frac{1}{2} \partial_{\mu} \Phi \partial^{\mu} \Phi - \frac{1}{2} M^2 \Phi^2 + \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{1}{2} m^2 \phi^2 - \mu \Phi \phi \phi$$

If $M > 2m$ then the last term means that the decay $\Phi \to \phi\phi$ is possible. Calculate the width $\Gamma$ for this decay to first order in $\mu$.

Hint: The interaction Hamiltonian is given by $\mathcal{H}_{int} = -\mathcal{L}_{int} = \mu \Phi \phi \phi$. With this you can calculate the invariant matrix-element

$$i\mathcal{M}(m_A \to p_1 p_2) (2\pi)^4 \delta^{(4)}(p_A - p_1 - p_2) = \langle \vec{p}_2 \vec{p}_1 | T\left\{ \exp \left[ -i \int_{-T}^{T} dt H_1(t) \right] \right\} |\vec{p}_A\rangle_0$$

Finally integrate $|\mathcal{M}|^2$ with the appropriate prefactors over the two-particle phase space.

Please send an email to Johan.Rathsman@thep.lu.se at least one hour before the problem solving session stating the problems that you are willing to solve on the blackboard and the problems that you would like to see the solution to. (Click on the email address to get a preformatted mail.)