Answers exam FYTB13, May 31, 2017

1. a) Gauss law: $\vec{\nabla}\cdot\vec{D} = \rho_f \Rightarrow \oint \vec{D}\cdot d\vec{S} = Q_{f,encl}$

Spherical symmetry $\Rightarrow$ $\vec{D} = \frac{Q}{4\pi r^2}\hat{r}$ for all r

b) $\vec{D} = \varepsilon_0 \varepsilon_r \vec{E} \Rightarrow$

$$\vec{E}_{r<a} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$$

$$\vec{E}_{a<r<b} = \frac{Q}{4\pi\varepsilon_0\varepsilon_r r^2}\hat{r}, \quad \varepsilon_r = 2$$

$$\vec{E}_{r>b} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$$

c) $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \Rightarrow \oint \vec{E}\cdot d\vec{S} = \frac{Q_{encl}}{\varepsilon_0}$

Gaussian pillbox $\Rightarrow (\vec{E}_{above} - \vec{E}_{below})\cdot\hat{r}\, dA = \frac{\sigma}{\varepsilon_0} dA$

$$\Rightarrow \sigma_{r=b} = \frac{Q}{4\pi b^2}\left(1 - \frac{1}{\varepsilon_r}\right) = \frac{1}{2}\frac{Q}{4\pi b^2}$$

$$\sigma_{r=a} = \frac{Q}{4\pi a^2}\left(\frac{1}{\varepsilon_r} - 1\right) = -\frac{1}{2}\frac{Q}{4\pi a^2}$$

Integration $\Rightarrow$ $\oint \sigma_{r=b}\, dS = \frac{Q}{2}$

$$\oint \sigma_{r=a}\, dS = -\frac{Q}{2}$$

d) By definition $\vec{E} = -\vec{\nabla}V$ (static field)

Spherical coord + symmetry $\Rightarrow$ $\vec{E} = -\hat{r}\frac{\partial}{\partial r}V$

$$\Rightarrow V_{r>b} = -\int_{\infty}^{r} \frac{Q}{4\pi\varepsilon_0 r'^2}dr' = \frac{Q}{4\pi\varepsilon_0 r}$$

$$V_{a<r<b} = \frac{Q}{4\pi\varepsilon_0 b} - \int_b^r \frac{Q}{4\pi\varepsilon_0\varepsilon_r r'^2}dr' = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{b} + \frac{1}{r\varepsilon_r} - \frac{1}{b\varepsilon_r}\right)$$

$$V_{r<a} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{b}+\frac{1}{\varepsilon_r}\left(\frac{1}{a}-\frac{1}{b}\right)\right) - \int_a^r \frac{Q}{4\pi\varepsilon_0 r'^2}\,dr' =$$

$$= \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{b}+\frac{1}{\varepsilon_r}\left(\frac{1}{a}-\frac{1}{b}\right)-\frac{1}{a}+\frac{1}{r}\right) =$$

$$= \frac{Q}{4\pi\varepsilon_0}\left(\left(\frac{1}{a}-\frac{1}{b}\right)\left(\frac{1}{\varepsilon_r}-1\right)+\frac{1}{r}\right)$$

e) $W = \frac{1}{2}\int \vec{E}\cdot\vec{D}\,d^3r$

$$\Rightarrow \Delta W = W_{with} - W_{without} = \frac{1}{2}\int\left(\vec{E}_{with}-\vec{E}_{without}\right)\cdot\vec{D}\,d^3r =$$

$$= \frac{1}{2}\frac{Q^2}{(4\pi)^2\varepsilon_0}\int_a^b\left(\frac{1}{r^2}\right)^2\left(\frac{1}{\varepsilon_r}-1\right)r^2dr\int d\Omega =$$

$$= \frac{Q^2}{8\pi\varepsilon_0}\left(\frac{1}{\varepsilon_r}-1\right)\left[-\frac{1}{r}\right]_a^b = \frac{Q^2}{8\pi\varepsilon_0}\underbrace{\left(\frac{1}{\varepsilon_r}-1\right)}_{<0}\underbrace{\left(\frac{1}{a}-\frac{1}{b}\right)}_{>0}$$

$\therefore\ \Delta W < 0$

f) ΔW negative means the configuration with the dielectric is energetically more favourable

2. a) magnetic flux $\Phi = \int \vec{B} \cdot d\vec{S} = [\vec{B} = \vec{\nabla} \times \vec{A}] =$

$$= \int (\vec{\nabla} \times \vec{A}) \cdot d\vec{S} = [\text{Stokes}] = \oint_C \vec{A} \cdot d\vec{l}$$

b) $$\vec{A}_1(\vec{r}_2) = \frac{\mu_0 I_1}{4\pi} \oint_{C_1} \frac{d\vec{l}_1}{|\vec{r}_2 - \vec{r}_1|}$$

$$\Rightarrow \Phi_{21} = \oint_{C_2} \vec{A}_1(\vec{r}_2) \cdot d\vec{l}_2 = I_1 \underbrace{\frac{\mu_0}{4\pi} \oint_{C_2} \oint_{C_1} \frac{d\vec{l}_1}{|\vec{r}_2 - \vec{r}_1|} \cdot d\vec{l}_2}_{M_{21}}$$

c) $$M_{12} = \frac{\mu_0}{4\pi} \oint_{C_1} \oint_{C_2} \frac{d\vec{l}_2}{|\vec{r}_1 - \vec{r}_2|} \cdot d\vec{l}_1 = [\text{change ord. of int.}] =$$

$$= \frac{\mu_0}{4\pi} \oint_{C_2} \oint_{C_1} \frac{d\vec{l}_1}{|\vec{r}_1 - \vec{r}_2|} \cdot d\vec{l}_2 = \left[|\vec{r}_1 - \vec{r}_2| = |\vec{r}_2 - \vec{r}_1| \right] = M_{21}$$

3 a) $\vec{M} = M\hat{z}$ for $r<b$

$$\text{ME}\begin{cases}\vec{\nabla}\cdot\vec{B}=0 & \left[\vec{H}=\frac{1}{\mu_0}\vec{B}-\vec{M}\right] \Rightarrow \vec{\nabla}\cdot\vec{H} = -\vec{\nabla}\cdot\vec{M}\\ \vec{\nabla}\times\vec{H} = \vec{J}_f + \frac{\partial}{\partial t}\vec{D} & \left[\vec{J}_f=0, \frac{\partial}{\partial t}\vec{D}=0\right] \Rightarrow \vec{\nabla}\times\vec{H}=0\end{cases}$$

$$\Rightarrow \vec{H} = -\vec{\nabla}\psi \quad , \quad \vec{\nabla}^2\psi = \vec{\nabla}\cdot\vec{M}$$

b) $\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \int \vec{\nabla}\cdot\vec{B}\, d^3r = 0$

Gaussian pillbox $\Rightarrow$ $\hat{r}\cdot(\vec{B}_{out} - \vec{B}_{in}) = 0$

$$\Rightarrow \hat{r}\cdot(\vec{H}_{out} - \vec{H}_{in}) = -\hat{r}\cdot(\underbrace{\vec{M}_{out}}_{=0} - \underbrace{\vec{M}_{in}}_{=M\hat{z}}) = M\cos\theta$$

$\hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$

c) $\vec{M} = \text{const} \Rightarrow \vec{\nabla}\cdot\vec{M} = 0 \Rightarrow \vec{\nabla}^2\psi = 0$

Symmetry $\Rightarrow$ general sol'n

$$\psi = \sum_l \left(A_l r^l + B_l \frac{1}{r^{l+1}}\right) P_l(\cos\theta)$$

$r>b$: $\psi \to 0$ as $r\to\infty$ $\Rightarrow$ $A_l^{out} = 0$ for all l

$r<b$: ψ finite as $r\to 0$ $\Rightarrow$ $B_l^{in} = 0$ — II —

$r=b$: ψ continuous

$$\sum_l A_l^{in} b^l P_l(\cos\theta) = \sum_l B_l^{out}\frac{1}{b^{l+1}} P_l(\cos\theta)$$

P_l orthonormal $\Rightarrow$ $A_l^{in} b^l = B_l^{out}\frac{1}{b^{l+1}}$

$r=b$: $\hat{r}\cdot\vec{H}$ discontinuous

$$-\frac{\partial}{\partial r}\psi_{r>b} + \frac{\partial}{\partial r}\psi_{r<b} = M\cos\theta$$

$$\Rightarrow \sum_l (l+1)\frac{B_l^{out}}{b^{l+2}} P_l(\cos\theta) + \sum_l l \frac{B_l^{out}}{b^{2l+1}} b^{l-1} P_l(\cos\theta) = M\cos\theta$$

P_l orthonormal $\Rightarrow B_l^{out} = 0$ for $l \neq 1$

$$B_1^{out}\left(\frac{2}{b^3} + \frac{1}{b^3}\right) = M \Rightarrow B_1^{out} = \frac{Mb^3}{3}$$

$$\Rightarrow A_1^{out} = \frac{M}{3}$$

$$\therefore \psi_{r>b} = \frac{Mb^3}{3r^2}\cos\theta \ , \ \psi_{r<b} = \frac{M}{3} r\cos\theta$$

d/ $\vec{H} = -\vec{\nabla}\psi \Rightarrow \begin{cases} \vec{H}_{r>b} = \frac{Mb^3}{3r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right) \\ \vec{H}_{r<b} = -\frac{1}{3}M\left(\hat{r}\cos\theta - \hat{\theta}\sin\theta\right) = -\frac{\vec{M}}{3} \end{cases}$

$\vec{B} = \mu_0(\vec{H} + \vec{M}) \Rightarrow$

$$\begin{cases} \vec{B}_{r>b} = \mu_0 \frac{Mb^3}{3r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right) \\ \vec{B}_{r<b} = \mu_0 \frac{2}{3}\vec{M} \end{cases}$$

4a) $\vec{\tilde{A}} = (\tilde{a}_x \hat{x} + \tilde{a}_y \hat{y} + \tilde{a}_z \hat{z}) e^{i(kx-\omega t)}$

$\tilde{V} = c\tilde{a}_x e^{i(kx-\omega t)}$

Lorenz gauge: $\frac{1}{c^2}\frac{\partial}{\partial t}\tilde{V} + \vec{\nabla}\cdot\vec{\tilde{A}} = 0$

$$\frac{1}{c^2}\frac{\partial}{\partial t}\tilde{V} = \frac{1}{c^2}(-i\omega)\, c\tilde{a}_x e^{i(kx-\omega t)}$$

$$\vec{\nabla}\cdot\vec{\tilde{A}} = ik\,\tilde{a}_x e^{i(kx-\omega t)}$$

$$\Rightarrow \frac{1}{c^2}\frac{\partial}{\partial t}\tilde{V} + \vec{\nabla}\cdot\vec{\tilde{A}} = \left(-i\frac{\omega}{c} + ik\right)\tilde{a}_x e^{i(kx-\omega t)} = 0$$

$$\frac{\omega}{c} = k$$

b) $\vec{\tilde{E}} = -\vec{\nabla}\tilde{V} - \frac{\partial}{\partial t}\vec{\tilde{A}} =$

$$= -c\tilde{a}_x\, ik\hat{x} e^{i(kx-\omega t)} + i\omega(\tilde{a}_x\hat{x} + \tilde{a}_y\hat{y} + \tilde{a}_z\hat{z})e^{i(kx-\omega t)}$$

$$= [kc = \omega] = i\omega(\tilde{a}_y\hat{y} + \tilde{a}_z\hat{z})e^{i(kx-\omega t)}$$

$$\vec{\tilde{B}} = \vec{\nabla}\times\vec{\tilde{A}} = i\vec{k}\times\vec{\tilde{A}} =$$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ k & 0 & 0 \\ \tilde{a}_x & \tilde{a}_y & \tilde{a}_z \end{vmatrix} e^{i(kx-\omega t)} = ik(-\tilde{a}_z\hat{y} + \tilde{a}_y\hat{z})e^{i(kx-\omega t)}$$

$$\vec{k} = k\hat{x}, \quad \vec{\tilde{B}}, \vec{\tilde{E}} \perp \hat{x} \Rightarrow \vec{\tilde{B}}\cdot\vec{k} = \vec{\tilde{E}}\cdot\vec{k} = 0$$

$$\vec{\tilde{E}}\cdot\vec{\tilde{B}} \propto (\tilde{a}_y\hat{y} + \tilde{a}_z\hat{z})\cdot(-\tilde{a}_z\hat{y} + \tilde{a}_y\hat{z}) = -\tilde{a}_y\tilde{a}_z + \tilde{a}_y\tilde{a}_z = 0$$

c) homogeneous wave-eqn

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right)\tilde{\vec{E}} = 0 , \quad \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right)\tilde{\vec{B}} = 0$$

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\tilde{\vec{E}} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(i\omega(\tilde{a}_y\hat{y} + \tilde{a}_z\hat{z})\,e^{i(kx-\omega t)}\right) =$$

$$= \frac{(-i\omega)^2}{c^2}\left(i\omega(\tilde{a}_y\hat{y} + \tilde{a}_z\hat{z})\,e^{i(kx-\omega t)}\right) = -\frac{\omega^2}{c^2}\tilde{\vec{E}}$$

$$\vec{\nabla}^2\tilde{\vec{E}} = \vec{\nabla}^2\left(i\omega(\tilde{a}_y\hat{y} + \tilde{a}_z\hat{z})\,e^{i(kx-\omega t)}\right) =$$

$$= (ik)^2\left(i\omega(\tilde{a}_y\hat{y} + \tilde{a}_z\hat{z})\,e^{i(kx-\omega t)}\right) = -k^2\tilde{\vec{E}}$$

$$= \left[k = \frac{\omega}{c}\right] = -\frac{\omega^2}{c^2}\tilde{\vec{E}} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\tilde{\vec{E}} \qquad \square$$

same way for $\tilde{\vec{B}}$

d) By def. $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$

Complex not. $\langle\vec{S}\rangle = \frac{1}{2\mu_0}\tilde{\vec{E}}\times\tilde{\vec{B}}^* =$

$$= \frac{1}{2\mu_0}\,-i^2\omega k\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ & \tilde{a}_y & \tilde{a}_z\\ & -\tilde{a}_z^* & \tilde{a}_y^*\end{vmatrix} = \frac{1}{2\mu_0}\omega k\,\hat{x}\left(|\tilde{a}_y|^2 + |\tilde{a}_z|^2\right)$$

e) Lorentz force: $\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$

$$\vec{v}\times\tilde{\vec{B}} = \begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ v & 0 & 0\\ 0 & -\tilde{a}_z & \tilde{a}_y\end{vmatrix} = -v(\tilde{a}_y\hat{y} + \tilde{a}_z\hat{z})$$

$$\Rightarrow \vec{F} = q\,\mathrm{Re}\left[\underbrace{\left(i\omega(\tilde{a}_y\hat{y} + \tilde{a}_z\hat{z}) - ivk(\tilde{a}_y\hat{y} + \tilde{a}_z\hat{z})\right)}_{=0 \text{ if } v = c}e^{i(kx-\omega t)}\right]$$