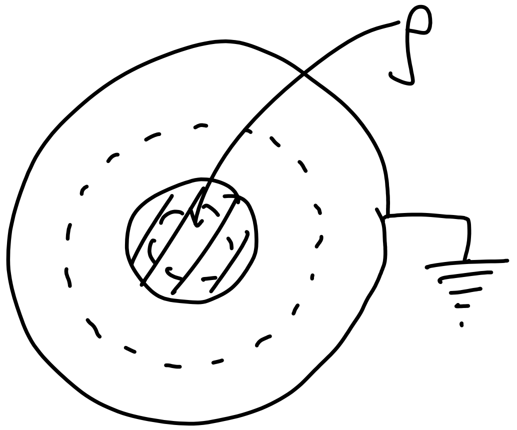
Answers to FYTB13 exam May 30, 2018
(i.e. not complete solutions)

1.



consider length l of cyl.

a) $s<a$: $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$

cyl. symmetry $\vec{E} = E\hat{s}$ and Gauss theorem

$$\int_{cyl} \vec{E}\cdot\hat{s}\, \underbrace{dS}_{s\,d\varphi\, dz} = \frac{\rho}{\varepsilon_0}\int \underbrace{d\tau}_{s\,ds\,d\varphi\, dz}$$

$$\Rightarrow E\, 2\pi l s = \frac{\rho}{\varepsilon_0}\frac{s^2}{2} 2\pi l \Rightarrow \vec{E} = \frac{\rho}{2\varepsilon_0} s\,\hat{s}$$

$a<s<b$: $\vec{\nabla}\cdot\vec{D} = \rho_f = \rho$, $\vec{D} = D\hat{s}$

similarly to above $\int \vec{D}\cdot\hat{s}\, dS = \rho\int d\tau$

$$\Rightarrow D\, 2\pi l s = \rho\,\frac{1}{2}a^2 2\pi l \Rightarrow \vec{D} = \frac{\rho}{2}\frac{a^2}{s}\hat{s}$$

linear medium: $\vec{D} = \varepsilon\vec{E} \Rightarrow \vec{E} = \frac{\rho}{2\varepsilon}\frac{a^2}{s}\hat{s}$

b) by def $\vec{E} = -\vec{\nabla}V$

cyl. coord $\vec{\nabla}V = \hat{s}\frac{\partial}{\partial s}V + \hat{\varphi}\frac{1}{s}\frac{\partial}{\partial\varphi}V + \hat{z}\frac{\partial}{\partial z}V$.

$$\Rightarrow E = -\frac{\partial}{\partial s}V, \quad V(s) = -\int_{s_0}^{s} E(s')\,ds'$$

shell grounded $\Rightarrow$ $V(b) = 0$

$a < s < b$: $V(s) = -\frac{\rho a^2}{2\varepsilon} \int_b^s \frac{1}{s'} ds' = \ldots = \frac{\rho a^2}{2\varepsilon} \ln \frac{b}{s}$

$s = a$: $V(a) = \frac{\rho a^2}{2\varepsilon} \ln \frac{b}{a}$

$s < a$: $V(s) = -\frac{\rho}{2\varepsilon_0} \int_a^s s' ds' + \frac{\rho a^2}{2\varepsilon} \ln \frac{b}{a} = \ldots =$

$= \frac{\rho}{4\varepsilon_0}(a^2 - s^2) + \frac{\rho a^2}{2\varepsilon} \ln \frac{b}{a}$

c) $V = 0$ for $s > b$ $\Rightarrow$ $\vec{E} = 0$ $\Rightarrow$ $\vec{D} = 0$ for $s > b$

Boundary cond $(\vec{D}_{out} - \vec{D}_{in}) \cdot \hat{s} = \sigma_f$

$\Rightarrow$ $\sigma_f(s = b) = -\frac{\rho a^2}{2b}$

alt. total enclosed free charge $= 0$

$\Rightarrow$ $Q_{rod} = \pi a^2 l \rho = -Q_{f,shell} = \sigma_f 2\pi b l$

d) linear med. $\Rightarrow$ $\vec{P} = \varepsilon_0 \chi_e \vec{E}$

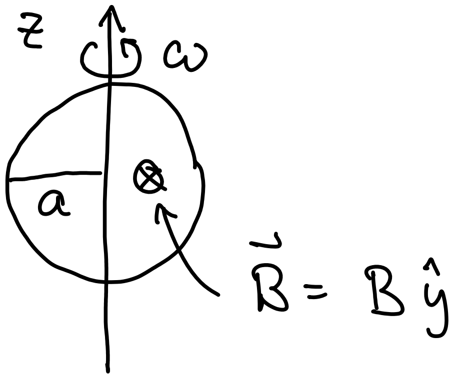
by def $\sigma_b = \vec{P} \cdot \hat{n}$

$s = b$: $\hat{n} = \hat{s}$ $\Rightarrow$ $\sigma_b = \frac{\chi_e}{\varepsilon_r} \frac{\rho a^2}{2b}$

$s = a$: $\hat{n} = -\hat{s}$ $\Rightarrow$ $\sigma_b = -\frac{\chi_e}{\varepsilon_r} \frac{\rho a}{2}$

check! $Q_b = 2\pi b l \frac{\chi_e}{\varepsilon_r} \frac{\rho a^2}{2b} - 2\pi a l \frac{\chi_e}{\varepsilon_r} \frac{\rho a}{2} = 0$

2.



a) $$\Phi = \int \vec{B}\cdot d\vec{S} = B\,\hat{y}\cdot\hat{\varphi}\,\pi a^2$$

$$\hat{\varphi} = -\hat{x}\sin(\omega t+\delta) + \hat{y}\cos(\omega t+\delta)$$

$$\hat{\varphi} = \hat{y} \quad \text{at} \quad t=0 \quad \Rightarrow \quad \delta = 0$$

$$\therefore \Phi = B\pi a^2 \cos(\omega t)$$

b) $$\mathcal{E} = -\frac{d}{dt}\Phi = +B\pi a^2 \omega \sin(\omega t)$$

c) $$I = \frac{\mathcal{E}}{R} = \frac{\omega B \pi a^2}{R}\sin(\omega t)$$

$$R = \frac{2\pi a}{\sigma A} \quad \Rightarrow \quad I = \frac{\omega B \sigma A a}{2}\sin(\omega t)$$

d) $$\vec{m} = I\int d\vec{S} = I\,\hat{\varphi}\,\pi a^2 =$$

$$= \frac{1}{2}\omega B \sigma A\, \pi a^3 \sin(\omega t)\left(-\hat{x}\sin(\omega t) + \hat{y}\cos(\omega t)\right)$$

$$= \frac{1}{4}\omega B \sigma A \pi a^3\left(+\hat{x}(\cos(2\omega t)-1) + \hat{y}\sin(2\omega t)\right)$$

$$\left[\cos(2\alpha) = 1 - 2\sin^2\alpha\right]$$

$$\langle\vec{m}\rangle = -\frac{1}{4}\omega B \sigma A \pi a^3\,\hat{x}$$

3. Standing wave

$$\vec{E}(r,t) = E_0 \, \hat{e} \cos(\vec{k}\cdot\vec{r}) \cos(\omega t)$$

a) $\cos(\alpha \pm \beta) = \cos\alpha \cos\beta \mp \sin\alpha \sin\beta$

$$\Rightarrow \cos\alpha \cos\beta = \frac{1}{2}\Big[\cos\alpha\cos\beta + \sin\alpha\sin\beta + \cos\alpha\cos\beta - \sin\alpha\sin\beta\Big]$$

$$= \frac{1}{2}\Big[\cos(\alpha-\beta) + \cos(\alpha+\beta)\Big]$$

$$\Rightarrow \vec{E}(\vec{r},t) = E_0 \hat{e}\Big[\cos(\vec{k}\cdot\vec{r} - \omega t) + \cos(\vec{k}\cdot\vec{r} + \omega t)\Big]$$

b) $\frac{\partial}{\partial t}\vec{B} = -\vec{\nabla}\times\vec{E} = \vec{k}\times\hat{e}\, E_0 \sin(\vec{k}\cdot\vec{r})\cos(\omega t)$

$$= \frac{\partial}{\partial t}\frac{1}{\omega}\vec{k}\times\hat{e}\, E_0 \sin(\vec{k}\cdot\vec{r})\sin(\omega t)$$

$$\Rightarrow \vec{B} = \big[\omega = kc\big] = \frac{1}{c}\hat{k}\times\hat{e}\, E_0 \sin(\vec{k}\cdot\vec{r})\sin(\omega t)$$

c) $\hat{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} =$

$$= \frac{1}{\mu_0}\frac{1}{c}\hat{e}\times(\hat{k}\times\hat{e})\, E_0^2 \cos(\vec{k}\cdot\vec{r})\sin(\vec{k}\cdot\vec{r})\cos(\omega t)\sin(\omega t)$$

$$= \frac{1}{4\mu_0 c}\hat{k}\sin(2\vec{k}\cdot\vec{r})\sin(2\omega t)$$

$\Rightarrow \langle \vec{S} \rangle = 0$ since $\langle \sin(2\omega t)\rangle = 0$

so there is no net transport - only back and forth

4. Vector potential (using complex notation)

$$\tilde{\vec{A}}(\vec{r},t) = \hat{x}\, A_0\, e^{i(kz-\omega t)}$$

axial gauge, $V=0$, A_0 real

a, by def

$$\tilde{\vec{E}} = -\frac{\partial}{\partial t}\tilde{\vec{A}} = +i\omega\, \hat{x}\, A_0\, e^{i(kz-\omega t)}$$

$$\tilde{\vec{B}} = \vec{\nabla}\times\tilde{\vec{A}} = ik\hat{z}\times\hat{x}\, A_0\, e^{i(kz-\omega t)}$$

physical fields:

$$\vec{E} = \mathrm{Re}(\tilde{\vec{E}}) = -\omega\,\hat{x}\, A_0 \sin(kz-\omega t)$$

$$\vec{B} = \mathrm{Re}(\tilde{\vec{B}}) = -k\,\hat{y}\, A_0 \sin(kz-\omega t)$$

b, Lorentz force:

$$\vec{F} = q(\vec{E}+\vec{v}\times\vec{B}), \quad k=\frac{\omega}{c}$$

$$\Rightarrow \vec{F} = -q\omega A_0 \sin(kz-\omega t)\left[\hat{x} + \frac{1}{c}\vec{v}\times\hat{y}\right]$$

c, $\vec{v} = v\hat{z}$, $z(t) = z_0 + vt$

$$\Rightarrow \vec{F} = -q\omega A_0 \sin(kz_0 + kvt - \omega t)\left(\hat{x} - \frac{v}{c}\hat{x}\right) =$$

$$= -q\omega A_0\, \hat{x}\left(1-\frac{v}{c}\right)\sin\left[kz_0 - \omega\left(1-\frac{v}{c}\right)t\right]$$

Force in $\pm\hat{x}$ direction. Acceleration sideways and fluctating in time, $\langle\vec{F}\rangle = 0$.
In addition the force vanishes as $v\to c$ and at the same time the fluctuations slow down.