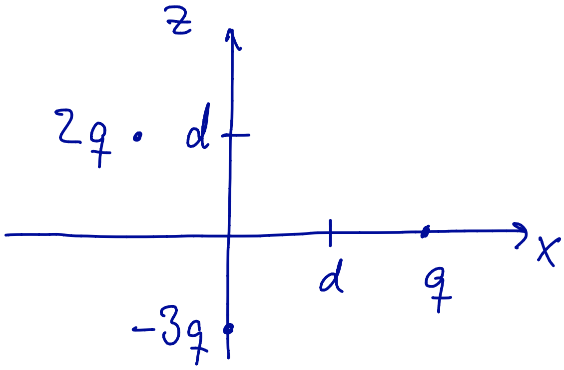
Answers to EM exam Nov 2, 2018
(note: not complete solutions)

1. a) total charge

$Q = 2q - 3q + q = 0$

$\Rightarrow$ monopole vanishes



b) dipole moment

$$\vec{p} = (-d\hat{x} + d\hat{z})2q + (-d\hat{z})(-3q) + 2d\hat{x}\,q =$$
$$= (-2q + 2q)\,d\hat{x} + (2q + 3q)\,d\hat{z} = 5qd\hat{z}$$

c) Inserting $\vec{p} = 5qd\hat{z}$ with $\hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$ gives

$$V_{dip}(\vec{r}) = \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0 r^3} = \frac{5qd\,r\cos\theta}{4\pi\varepsilon_0 r^3} = \frac{5qd\cos\theta}{4\pi\varepsilon_0 r^2}$$

d) V for collection of point charges

$$V(\vec{r}) = \sum_i \frac{q_i}{4\pi\varepsilon_0|\vec{r}-\vec{r}_i|} =$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{-3q}{|\vec{r}+d\hat{z}|} + \frac{q}{|\vec{r}-2d\hat{x}|} + \frac{2q}{|\vec{r}+d\hat{x}-d\hat{z}|}\right]$$

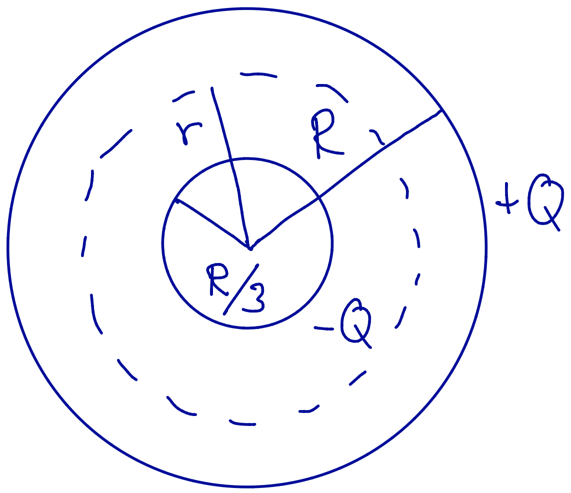
$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{-3q}{r}\left(1 - \frac{\vec{r}\cdot d\hat{z}}{r^2}\right) + \frac{q}{r}\left(1 + \frac{2\vec{r}\cdot d\hat{x}}{r^2}\right) + \right.$$
$$\left. + \frac{2q}{r}\left(1 - \frac{\vec{r}\cdot(d\hat{x} - d\hat{z})}{r^2}\right) + \dots\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{(-3+1+2)q}{r} + \frac{qd}{r^2}\,\hat{r}\cdot\left(3\hat{z} + 2\hat{x} - 2\hat{x} + 2\hat{z}\right) + \dots\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{r}\cdot 0 + \frac{qd}{r^2}\, 5\,\hat{r}\cdot\hat{z} + \dots\right] = \frac{5qd\cos\theta}{r^2} + \dots$$

$$= V_{dip}(\vec{r}) + \dots$$

2.



a) Gauss' law, $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$

Integrate over spherical volume with radius r

$$\int_V \vec{\nabla}\cdot\vec{E}\, d\tau = \frac{1}{\varepsilon_0}\int_V \rho\, d\tau$$

Gauss theorem

$$\int_{\partial V} \vec{E}\cdot d\vec{S} = \frac{1}{\varepsilon_0}\int_V \rho\, d\tau$$

Spherical symmetry $\vec{E} = E(r)\,\hat{r}$

$\frac{R}{3} < r < R$: $\int E(r)\,\hat{r}\cdot\hat{r}\, r^2 d\varphi \sin\theta\, d\theta = -\frac{1}{\varepsilon_0} Q$

$$\Rightarrow 4\pi r^2 E(r) = -\frac{1}{\varepsilon_0} Q$$

$$\Rightarrow \vec{E} = -\frac{Q}{4\pi\varepsilon_0 r^2}\,\hat{r}$$

In the same way $\vec{E}=0$ for $r<R/3$ and $r>R$

The potential V:

$$\vec{E} = -\vec{\nabla} V \Rightarrow V(r) = -\int_{\infty}^{r} \vec{E}\cdot d\vec{l},$$

$\frac{R}{3} \leq r \leq R$: $\quad V(r) = +\int_{R/3}^{r} \frac{Q}{4\pi\varepsilon_0 r'^2}\hat{r}\cdot\hat{r}\,dr' =$

$[V(\frac{R}{3})=0]$

$$= \frac{Q}{4\pi\varepsilon_0}\left[-\frac{1}{r'}\right]_{R/3}^{r} = \frac{Q}{4\pi\varepsilon_0}\left[\frac{3}{R}-\frac{1}{r}\right]$$

$\Rightarrow$ potential difference, $U = V(R)-V(\frac{R}{3}) = \frac{Q}{4\pi\varepsilon_0}\frac{2}{R}$

By definition $C = \frac{Q}{U} \Rightarrow C_0 = 2\pi\varepsilon_0 R$

b) The $\vec{D}$-field: $\quad \vec{\nabla}\cdot\vec{D} = \rho_f$

In the same way as before we get

$\frac{R}{3} < r < R$: $\quad \vec{D} = -\frac{Q}{4\pi r^2}\hat{r}$

I: region $\frac{R}{3} < r < \frac{R}{2}$ filled

$$\Rightarrow E_{R/3<r<R/2} = -\frac{Q}{4\pi\varepsilon r^2}\hat{r}, \quad V(\frac{R}{2}) = \frac{Q}{4\pi\varepsilon}\left(\frac{3}{R}-\frac{2}{R}\right)$$

$$E_{R/2<r<R} = -\frac{Q}{4\pi\varepsilon_0 r^2}\hat{r},$$

$$V(R) = V(\frac{R}{2}) + \frac{Q}{4\pi\varepsilon_0}\left(\frac{2}{R}-\frac{1}{R}\right)$$

$$\Rightarrow \quad U_I = \frac{Q}{4\pi\varepsilon_0}\frac{1}{R}\left(\frac{1}{\varepsilon_r}+1\right) = \frac{Q}{4\pi\varepsilon_0}\frac{\varepsilon_r+1}{\varepsilon_r}\frac{1}{R}$$

and $C_I = \frac{Q}{U_I} = 4\pi\varepsilon_0 R \frac{\varepsilon_r}{\varepsilon_r+1} = \frac{2\varepsilon_r}{\varepsilon_r+1} C_0$

II: region $R/2 < r < R$ filled

again $\vec{D} = -\frac{Q}{4\pi r^2}\hat{r}$

$$\Rightarrow \vec{E}_{R/3<r<R/2} = -\frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}, \quad V\left(\frac{R}{2}\right) = \frac{Q}{4\pi\varepsilon_0}\left(\frac{3}{R}-\frac{2}{R}\right)$$

$$\vec{E}_{R/2<r<R} = -\frac{Q}{4\pi\varepsilon r^2}\hat{r}$$

$$V(R) = V\left(\frac{R}{2}\right) + \frac{Q}{4\pi\varepsilon}\left(\frac{2}{R}-\frac{1}{R}\right)$$

$$\Rightarrow U_{II} = \frac{Q}{4\pi\varepsilon_0}\frac{1}{R}\left(1+\frac{1}{\varepsilon_r}\right) = U_I \Rightarrow C_{II} = C_I$$

c) By definition $\vec{D} = \varepsilon_0\vec{E} + \vec{P} = \varepsilon_0\varepsilon_r\vec{E}$

linear medium $\vec{P} = \chi_e\varepsilon_0\vec{E} = (\varepsilon_r - 1)\varepsilon_0\vec{E}$

know that $\sigma_b = \vec{P}\cdot\hat{n}$ and $\rho_b = -\vec{\nabla}\cdot\vec{P} = 0$

since $\vec{\nabla}\cdot\vec{E} = 0$ in med.

I: $r = R/3$, $\hat{n} = -\hat{r}$, $\vec{E} = -\frac{9Q}{4\pi\varepsilon R^2}\hat{r}$

$$\Rightarrow \sigma_b = (\varepsilon_r - 1)\varepsilon_0\frac{9Q}{4\pi\varepsilon_0\varepsilon_r R^2}, \quad Q_b = \frac{\varepsilon_r-1}{\varepsilon_r}Q$$

$r = R/2$, $\hat{n} = \hat{r}$, $\vec{E} = -\frac{4Q}{4\pi\varepsilon R^2}\hat{r}$

$$\Rightarrow \sigma_b = -(\varepsilon_r - 1)\varepsilon_0\frac{4Q}{4\pi\varepsilon_0\varepsilon_r R^2}, \quad Q_b = -\frac{\varepsilon_r-1}{\varepsilon_r}Q$$

II: $r = R/2, \quad \hat{n} = -\hat{r}, \quad \vec{E} = -\frac{4Q}{4\pi \varepsilon R^2}\hat{r}$

$$\Rightarrow \sigma_b = (\varepsilon_r - 1)\varepsilon_0 \frac{4Q}{4\pi\varepsilon R^2}, \quad Q_b = \frac{\varepsilon_r - 1}{\varepsilon_r} Q$$

$r = R, \quad \hat{n} = \hat{r}, \quad \vec{E} = -\frac{Q}{4\pi\varepsilon R^2}\hat{r}$

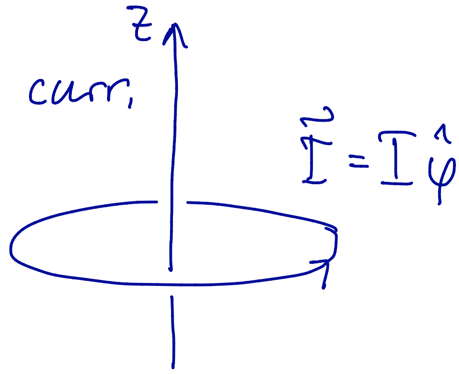
$$\Rightarrow \sigma_b = -(\varepsilon_r - 1)\varepsilon_0 \frac{Q}{4\pi\varepsilon_0\varepsilon_r R^2}, \quad Q_b = -\frac{\varepsilon_r - 1}{\varepsilon_r} Q$$

d) region I has volume $\frac{4\pi}{3}\left(\left(\frac{R}{2}\right)^3 - \left(\frac{R}{3}\right)^3\right) = \frac{4\pi R^3}{3}\left(\frac{1}{8} - \frac{1}{27}\right)$

" II " " $\frac{4\pi}{3}\left(R^3 - \left(\frac{R}{2}\right)^3\right) = \frac{4\pi R^3}{3}\left(1 - \frac{1}{8}\right)$

$\Rightarrow$ cheaper to fill region I !

3. a) Biot-Savart law for line curr.



$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{I}(\vec{r}') \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} dl'$$

want field on axis, $\vec{r} = z\hat{z}$

circular ring $\vec{r}' = R\hat{s}, \quad dl' = R\,d\varphi$

$$\Rightarrow \vec{B}(z) = \frac{\mu_0}{4\pi}\int \frac{I\hat{\varphi} \times (z\hat{z} - R\hat{s})}{|z\hat{z} - R\hat{s}|^3} R\,d\varphi$$

$$= \frac{\mu_0 I}{4\pi}\int \frac{\cancelto{0}{z\hat{s}} + R\hat{z}}{(z^2 + R^2)^{3/2}} R\,d\varphi = \frac{\mu_0 I R^2}{2(z^2 + R^2)^{3/2}}\hat{z}$$

$$\int_0^{2\pi} \hat{s}\,d\varphi = \int_0^{2\pi} (\hat{x}\cos\varphi + \hat{y}\sin\varphi)\,d\varphi = 0 \quad \text{(from symm.)}$$

b) Given $\vec{A}_{dip} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2}$, $\vec{m} = m\hat{z}$

$$\Rightarrow \vec{A}_{dip} = \frac{\mu_0 m}{4\pi}\frac{\hat{z}\times\hat{r}}{r^2} \underset{\uparrow}{=} \frac{\mu_0 m}{4\pi}\frac{\sin\theta}{r^2}\hat{\varphi}$$

$$\hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$$

by definition

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{\mu_0 m}{4\pi}\underbrace{\vec{\nabla}\times\left(\frac{\sin\theta}{r^2}\hat{\varphi}\right)}$$

$$\hat{r}\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\sin^2\theta}{r^2}\right) + \hat{\theta}\left(-\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\sin\theta}{r}\right)\right)$$

$$= \frac{\mu_0 m}{4\pi}\left(\hat{r}\frac{2\cos\theta}{r^3} + \hat{\theta}\frac{\sin\theta}{r^3}\right) \qquad \square$$

c) Far away from the loop, $|z| \gg R$, the field is that of a perfect dipole.

i) $z\to+\infty$, $\theta=0$, $\cos\theta=1$, $\hat{r}=\hat{z}$, $r=z$:

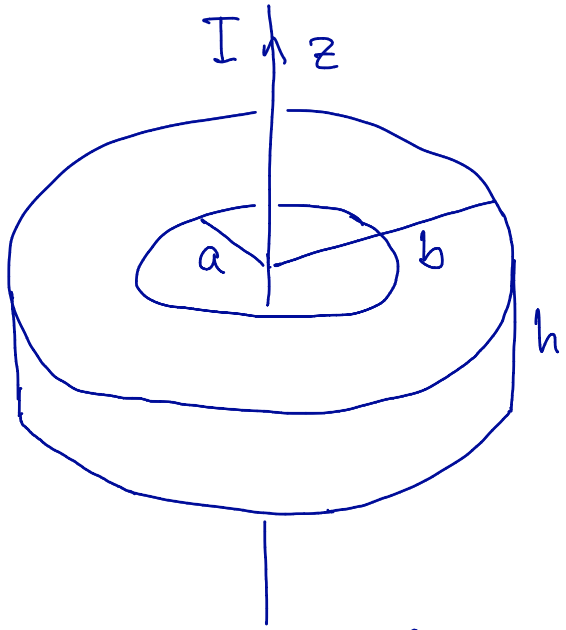
$$\vec{B}_{dip}\to\frac{\mu_0 m}{4\pi}\frac{2}{z^3}\hat{z}, \quad \vec{B}_{loop}\to\frac{\mu_0 I R^2}{2z^3}\hat{z} \Rightarrow \vec{m} = I\pi R^2\hat{z}$$

ii) $z\to-\infty$, $\theta=\pi$, $\cos\theta=-1$, $\hat{r}=-\hat{z}$, $r=|z|$;

$$\vec{B}_{dip}\to\frac{\mu_0 m}{4\pi}\frac{-2}{|z|^3}(-\hat{z}), \quad \vec{B}_{loop}\to\frac{\mu_0 I R^2}{2|z|^3}\hat{z} \Rightarrow \vec{m} = I\pi R^2\hat{z}$$

In both limits we find $\vec{m} = I\pi R^2\hat{z}$

4.



a) Ampere's law for steady current

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

integrate over surface perp to I with radius s

$$\int_S (\vec{\nabla} \times \vec{B}) \cdot d\vec{S} = \mu_0 \int \vec{J} \cdot d\vec{S} = \mu_0 I$$

Stokes'

right-hand rule + symmetry

$$\oint_{\partial S} \vec{B} \cdot d\vec{l} = \left[\vec{B} = B(s)\hat{\varphi} \, , \; d\vec{l} = \hat{\varphi} \, s \, d\varphi \right] =$$

$$= \int_0^{2\pi} B(s) \, s \, d\varphi = 2\pi s \, B(s)$$

$$\Rightarrow \quad B(s) = \frac{\mu_0 I}{2\pi s} \quad , \quad \vec{B} = \frac{\mu_0 I}{2\pi s} \hat{\varphi}$$

b) Magnetic flux

$$\Phi_{1\text{-turn}} = \oint_{1\text{-turn}} \vec{B} \cdot \underbrace{d\vec{S}}_{\hat{\varphi}\, ds\, dz} = \frac{\mu_0 I}{2\pi} \int_a^b \int_0^h \frac{1}{s} \, ds \, dz = \frac{\mu_0 I}{2\pi} h \ln \frac{b}{a}$$

$$\Rightarrow \quad \Phi_{N\text{-turns}} = N \, \Phi_{1\text{-turn}} = \frac{\mu_0 N I}{2\pi} h \ln \frac{b}{a}$$

c) assume $I = I_0 \sin(\omega t)$

quasi-static approx. $\Rightarrow$ $\vec{B} = \frac{\mu_0 I_0}{2\pi s} \sin(\omega t)\, \hat{\varphi} = B_\varphi \hat{\varphi}$

Faraday's law: $\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t} \vec{B}$

integrate over cross-sectional area of coil

$$\underbrace{\int_S (\vec{\nabla} \times \vec{E}) \cdot d\vec{S}}_{\oint \vec{E} \cdot d\vec{l} \,=\, \mathcal{E}} = -\int_S \frac{\partial}{\partial t} \vec{B} \cdot d\vec{S} = -\frac{d}{dt} \Phi$$

$$\Rightarrow \quad \mathcal{E} = -\frac{\partial}{\partial t} \Phi_N = -\frac{\mu_0 N I_0 h}{2\pi} \ln\frac{b}{a}\, \omega \cos(\omega t)$$

d) mutual inductance M defined by $\mathcal{E} = -M \frac{dI}{dt}$

$$\Rightarrow \quad M = \frac{\mathcal{E}}{-\frac{dI}{dt}} = \left[\frac{dI}{dt} = \omega I_0 \cos(\omega t)\right] = \frac{\mu_0 N h}{2\pi} \ln\frac{b}{a}$$

e) Faraday's law $\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t} \vec{B} = -\frac{\partial}{\partial t} B_\varphi \hat{\varphi}$

use cylindrical coord. $\vec{E} = E_s \hat{s} + E_\varphi \hat{\varphi} + E_z \hat{z}$

$$\Rightarrow \quad \vec{\nabla} \times \vec{E} = \hat{\varphi} \left[\frac{\partial}{\partial z} E_s - \frac{\partial}{\partial s} E_z\right] = -\frac{\partial}{\partial t} B_\varphi \hat{\varphi}$$

together with $\vec{\nabla} \cdot \vec{E} = 0$ this gives $E_s = 0$

(right-hand rule, same way as Biot-Savart for an infinite coil)

zero-point

$$\Rightarrow \quad E_z = \int \frac{\partial}{\partial t} B_\varphi \, ds = \frac{\omega \mu_0 I_0}{2\pi} \cos(\omega t) \ln\frac{b}{s}$$