Written exam, FYTB13/FYTA12, Electromagnetism, Nov 2, 2018, 08.00–13.00.

Allowed material: (a) one A4-sheet with notes; (b) pens, erasers, rulers and similar tools for drawing; (c) something to drink, eat, snacks, pillows, towels and similar necessities.

Total of 30 points, 15 will be required to pass.

Note! The problems are not ordered in difficulty. Read the text of each problem carefully before starting to solve it and make sure to motivate all steps and assumptions carefully.

Use separate sheets for each problem, mark them with your ID number, and write on only one side.

The **results** will be displayed in the theoretical physics corridor as soon as they are ready.

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1. *A not-so-ideal electric dipole* [6p]

A charge distribution ρ is made up of three point charges, $-3q$, q and $2q$, that are placed at, respectively, $-d\,\hat{\mathbf{z}}$, $2d\,\hat{\mathbf{x}}$, and $-d\,\hat{\mathbf{x}} + d\,\hat{\mathbf{z}}$, where d is a length scale.

a) [1p] Show that the monopole moment $\int \rho(\mathbf{r})\mathrm{d}\tau$ vanishes.

b) [2p] Determine the dipole moment $\mathbf{p} = \int \mathbf{r}\rho(\mathbf{r})\mathrm{d}\tau$.

c) [1p] Use the result from (***b***) to calculate the electric potential in the dipole approximation,

$$V_{\text{dip}}(\mathbf{r}) = \frac{\mathbf{p}\cdot\mathbf{r}}{4\pi\varepsilon_0 r^3}.$$

d) [2p] Calculate the exact $V(\mathbf{r})$, and show that the dipole approximation is correct to leading order in d/r. (*Hint:* You can use that $1/|\mathbf{r}-\mathbf{r}'| = 1/r + \mathbf{r}\cdot\mathbf{r}'/r^3 + \ldots$ if $r \gg r'$).

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2. *Poor man's capacitor* [9p]

A spherical capacitor is formed by a pair of conductors in the form of two concentric spherical shells, of respective radii $R/3$ and R.

a) [3p] Determine the resulting capacitance C_0, by assuming that the conductors at $r = R/3$ and $r = R$ carry the respective charges $-Q$ and $+Q$ as uniform surface charges. (*Hint:* Use e.g. the symmetry of the setup and Gauss' law to determine the electric field in the region between the terminals, and hence find the potential difference $U = \Delta V$. Motivate all statements, and specify any Gauss surfaces used!)

Now we wish to increase the capacitance using a dielectric with the relative permittivity $\varepsilon_r > 1$. To that end we consider the space between the conductors to be divided in two concentric regions: region I ($R/3 < r < R/2$) and region II ($R/2 < r < R$), and we wish to fill one of them with the dielectric.

b) [3p] Calculate the resulting capacitance for the two cases, and show that they both equal $C_0\frac{2\varepsilon_r}{\varepsilon_r+1}$. (*Hint:* In each case, first calculate the $\mathbf{D}$ field, and then the $\mathbf{E}$ field.)

c) [2p] Calculate the resulting total bound charge on each surface of the dielectric, for the two cases.

d) [1p] Assuming the dielectric to be expensive, which region would be the better choice to fill?

3. *A not-so-perfect magnetic dipole* [6p]

In one of the hand-in exercises you used the magnetic field from a circular loop with radius R carrying a current I. Here we want to compare that to the field from a perfect dipole.

a) [2p] Calculate the magnetic field (e.g. using Biot-Savart's law) on the symmetry axis (assumed to be in the z-direction) of the loop, and show that it is given by

$$\mathbf{B}_{\text{loop}}(z) = \frac{\mu_0 I}{2} \frac{R^2}{(R^2 + z^2)^{3/2}} \hat{\mathbf{z}}.$$

b) [2p] The vector potential of a perfect dipole is given by

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2}.$$

Use this to show that the magnetic field from a perfect dipole with $\mathbf{m} = m\,\hat{\mathbf{z}}$ is given by

$$\mathbf{B}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0 m}{4\pi r^3} \left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\theta}\right).$$

c) [2p] Deduce the dipole moment of the loop in (***a***), by comparing the results from (***a***) and (***b***) on the positive and negative z axis, respectively, in the appropriate limits.

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4. *The wire* [9p]

Consider an infinitely long wire, carrying a steady current I, running along the symmetry axis of a toroidal coil. The coil has a rectangular cross-section with inner radius a, outer radius b and height h.

a) [2p] Starting from Maxwell's equations on differential form, calculate the magnetic field.

b) [2p] Use the results from (***a***) to calculate the magnetic flux through the coil assuming that the coil has N turns. If you have not solved (***a***) you can use $\mathbf{B} = C_1\hat{\varphi}/s$.

c) [2p] Now assume that the current has a time-dependence $I = I_0 \sin(\omega t)$ and that the quasi-static approximation can be used. Calculate the induced emf in the coil. If you have not solved (***b***) you can use $\Phi = C_2 \ln(b/a)\sin(\omega t)$.

d) [1p] Calculate the mutual inductance between the wire and the coil.

e) [2p] Calculate the Faraday-induced electric field, surrounding the wire with the oscillating current from (***c***). Assume that the field vanishes for $s = b$.

——————— *Good Luck!* ———————