# Written exam, FYTB13/FYTA12, Electromagnetism, Oct 27, 2017, 14.00–19.00.

**Allowed material**: (a) one A4-sheet with notes; (b) pens, erasers, rulers and similar tools for drawing; (c) something to drink, eat, snacks, pillows, towels and similar necessities.

**Total of 30 points, 15 will be required to pass.**

Note! The problems are not ordered in difficulty. Read the text of each problem carefully before starting to solve it and make sure to motivate all steps and assumptions carefully.

Use separate sheets for each problem, mark them with your ID number, and write on only one side.

The **results** will be displayed in the theoretical physics corridor as soon as they are ready and at the latest on Friday Nov 10.

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## 1. ***Shelled*** (9p)

Consider three concentric metallic shells with radius $R_1 > R_2 > R_3$ respectively. The outermost shell has charge $q_1$, the middle one $q_2$ and the smallest one $q_3$.

***a***) [2p] Starting from Maxwell's equations, calculate the scalar potential $V$ as a function of $r$, the distance from the origin, in the regions i) ($r \geq R_1$), ii) ($R_1 > r \geq R_2$), iii) ($R_2 > r \geq R_3$), iv) ($r < R_3$).

***b***) [2p] Suppose that the innermost and outermost spheres are connected by a very thin metallic wire. What are the charges in the three spheres now?

***c***) [2p] Alternatively, suppose that we fill the region $R_1 > r > R_2$ with a linear dielectric material with permittivity $\epsilon_1$. Show that in the region ii) the potential can be written as:

$$\frac{q_1}{4\pi\epsilon_0 R_1} + \frac{(q_2+q_3)}{4\pi\epsilon_1}\left(\frac{1}{r} + \frac{\chi_e}{R_1}\right)$$

***d***) [3p] With the dielectric in place, what is the bound surface charge just inside the sphere with radius $R_1$ ($\sigma_{b,1}$) and just outside the sphere with radius $R_2$ ($\sigma_{b,2}$)? Using the total enclosed charge and Maxwell's equations, calculate $\mathbf{E}$ in region ii). Show that it agrees with the expression for $\mathbf{E}$ you would get by first calculating $\mathbf{D}$.

## 2. ***Having a spin*** (5p)

Consider a uniformly charged solid sphere (i.e. a ball) with total charge Q and radius $a$. The ball is spinning around an axis through its center with a constant angular velocity $\omega$.

***a***) [2p] In one of the hand-in exercises you have found that the vector potential from a spherical shell with radius $R$ and uniform surface charge $\sigma$ spinning around an axis through its center with an angular velocity $\omega$ is given by

$$\mathbf{A}_{\text{shell}} = \frac{\mu_0\omega\sigma R^4}{3}\frac{\sin\theta}{r^2}\hat{\varphi}$$

Use this result together with the superposition principle to show that the vector potential for the ball is given by

$$\mathbf{A}_{\text{ball}} = \frac{\mu_0\omega Q a^2}{20\pi}\frac{\sin\theta}{r^2}\hat{\varphi}$$

(Hint. Use that in this case $\sigma = \rho dR$)

***b***) [2p] Use the results from (***a***) to calculate the magnetic field $\mathbf{B}$ outside the ball.

***c***) [1p] What is the magnetic dipole moment of the ball? (Note that you can do this without any calculation.) Are there any higher multipole moments?

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**3. *Energy conservation*** (8p)
Consider two very long concentric solenoids along the same direction with radius $a$ and $b$ (with $b > a$) respectively. The magnitude of the current ($I$) and number of turns per unit length ($n$) is the same for both of them but the currents are in opposite direction. For definiteness you should assume that the current in the outer one is in the $\varphi$-direction with the solenoid being along the $z$-axis.

***a***) [3p] Calculate the magnetic field in the three regions $s > b$, $a < s < b$ and $s < a$. Argue carefully from first principles. You can use that $\mathbf{B} \to 0$ as $s \to \infty$.

***b***) [2p] Now assume that the current $I$ is time varying with $I = I_0 \sin(\omega t)$ and $\omega \ll c/b$ (with $c$ being the speed of light) such that the quasi-static approximation can be used. Calculate the induced electric field in the two regions $a < s < b$ and $s > b$ using the fact that it is in the $\varphi$-direction due to symmetry. If you have not solved (***a***) you can use $\mathbf{B} = C\sin(\omega t)\hat{\mathbf{z}}$ for $a < s < b$ and zero otherwise.

***c***) [1p] Show that the Poynting vector in the region $a < s < b$ is given by:

$$\mathbf{S}_{a<s<b} = -\frac{\mu_0 n^2 I_0^2}{4}\omega \sin(2\omega t)\frac{s^2 - a^2}{s}\hat{\mathbf{s}}$$

If you have not solved (***b***) you can use $\mathbf{E} = -C\omega\cos(\omega t)\dfrac{s^2 - a^2}{2s}\hat{\varphi}$ for $a < s < b$.

***d***) [2p] Calculate the energy density of the electromagnetic fields in the region $a < s < b$ and use this together with the results from (***c***) to verify that the electromagnetic energy is conserved locally (that is for any point in the region between the two solenoids) when taking into account that the quasi-static approximation has been used (i.e. when terms of the order $\omega a/c$ and $\omega b/c$ are neglected).

**4. *Tunnel Vision*** (8p)

In this problem we will work with plane wave solutions to Maxwell's equations. Using complex notation, the electric field is given by

$$\tilde{\mathbf{E}}(\mathbf{r}, t) = \tilde{\mathbf{E}}_0\, e^{\,i\,(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

with $\tilde{\mathbf{E}}_0 = \tilde{\mathrm{E}}\,\hat{\mathbf{e}}_i$.

***a***) [1p] Using Maxwell's equations, show that

$$\tilde{\mathbf{B}}(\mathbf{r}, t) = \tilde{\mathbf{B}}_0\, e^{\,i\,(\mathbf{k}\cdot\mathbf{r} - \omega t)} \quad \text{with } \tilde{\mathbf{B}}_0 = \frac{\tilde{\mathrm{E}}}{c}\,(\hat{\mathbf{k}} \times \hat{\mathbf{e}}_i)$$

***b***) [2p] Plug both expressions into Maxwell's equations in vacuum. Write them in terms of the wave vector $\mathbf{k}$ and $\hat{\mathbf{e}}_i$. How are $\mathbf{k}$ and $\hat{\mathbf{e}}_i$ related?

***c***) [1p] What is the meaning of the direction of the wave vector $\mathbf{k}$? How are $k$ and $\omega$ related to the speed of propagation of the wave?

Suppose the $xy$ plane forms the boundary between two "semi"-infinite linear media with indices of refraction $n_1$ and $n_2$ and take $n_2 > n_1$, $\mu_1 = \mu_2 = 1$. Consider an EM wave traveling in the $xz$ plane with Transverse Electric (TE) polarization from the first to the second medium and hitting the boundary at an incidence angle $\theta_i$ with respect to the $z$-axis. We would like to choose materials such that waves with incidence angles of $\theta_i < \pi/4$ loose at most half of their intensity while crossing it. In other words:

***d***) [3p] Find the value of $n_1/n_2$ such that for incidence angles smaller than $\pi/4$, the power per unit area of the reflected wave is **at most** half of the power per unit area of the incoming wave. Use the fact that for TE polarization, Fresnel equations read

$$\frac{\tilde{\mathrm{E}}_t}{\tilde{\mathrm{E}}_i} = \frac{2\mu_2 n_1 \cos\theta_i}{\mu_2 n_1 \cos\theta_i + \mu_1 n_2 \cos\theta_t} \qquad \frac{\tilde{\mathrm{E}}_r}{\tilde{\mathrm{E}}_i} = \frac{\mu_2 n_1 \cos\theta_i - \mu_1 n_2 \cos\theta_t}{\mu_2 n_1 \cos\theta_i + \mu_1 n_2 \cos\theta_t},$$

and the fact that both ratios are monotonic functions in the case of TE polarization. (Hint: You can use the fact that the intensity $I = |\langle \mathbf{S} \rangle \cdot \hat{z}|$ and the fact that the incoming and reflected waves can be treated separately in the case of plane waves. It is also very useful to notice that for $\theta_i = \pi/4$ one can write $\sin\alpha = \cos\alpha$ and also remember Snell's law).

***e***) [1p] Can you think of an example where finding such material could be useful?

——————— *Good Luck!* ———————