Invitation:

Electromagnetism is about finding out how electric charges/currents (charges in motion) affects other charges/currents. We do this by introducing electric ($\vec{E}$) and magnetic ($\vec{B}$) fields which depend on each other and the sources ρ (charge density) and $\vec{J}$ (current density) through

Maxwell's equations

$$
\begin{cases}
\vec{\nabla} \cdot \vec{E} = \frac{1}{\varepsilon_0} \rho & \text{(Gauss' law)} \\
\vec{\nabla} \cdot \vec{B} = 0 & \\
\vec{\nabla} \times \vec{E} + \frac{\partial}{\partial t} \vec{B} = 0 & \text{(Faraday's law)} \\
\vec{\nabla} \times \vec{B} - \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \vec{E} = \mu_0 \vec{J} & \text{(Ampère's law with Maxwell's correction)}
\end{cases}
$$

where ρ and $\vec{J}$ fulfill the continuity equation

$$\frac{\partial}{\partial t} \rho = -\vec{\nabla} \cdot \vec{J} \qquad \text{(follows from Gauss' and Ampère)}$$

The force on other charges/currents is then given by the Lorentz force law

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

which tells us how charges/currents move

Note that in general the e.m. fields, $\vec{E}$ and $\vec{B}$ as well as the density ρ and current $\vec{J}$ are functions of both the position $\vec{r}$ and time t - i.e. they are fields. By convention we do not write out this dependence.

We will solve Maxwell's equns using symmetry arguments as well as direct calculations using vector analysis.

The starting point will be the mathematics we need, which is given in chapter 1 and appendices A and B in the book by Griffiths. The material can also be found in ch 15 and 16 in the Calculus book by Adams and Essex.

In this course we will follow the tradition and start with considering the case when there is no time-dependence

$$\frac{\partial}{\partial t}\left(\vec{E}, \vec{B}, \rho, \vec{J}\right) = 0$$

and start by exploring <u>electrostatics</u> followed by <u>magnetostatics</u>. Only then will we include the time-dependence and study induction and waves

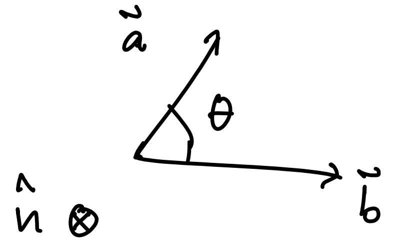
Finally, before starting we should also make it clear that you have already encountered many of the ideas and concepts of the course in earlier courses (Physics 1) – the difference is that now we will use much more rigor and mathematical tools that will allow us to solve much more complicated problems.

Mathematical tools

In order to proceed we need to develop some mathematical tools called vector calculus - how to take derivatives of vector fields and how these can be integrated. In other words the meaning of $\vec{\nabla}\cdot\vec{E}$ and $\vec{\nabla}\times\vec{E}$.

Let's start with a brief reminder about vectors which we will denote $\vec{a}, \vec{b}, \vec{c}$ etc. Note that they in principle could be vector <u>fields</u>, such as the electric and magnetic fields, which depend on the space-time point (x,y,z,t)

Consider two vectors $\vec{a}$ and $\vec{b}$ with an angle θ in between the directions



scalar product: $\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$
$= a b \cos\theta$

cross product: $\vec{a}\times\vec{b} = ab\sin\theta\ \hat{n}$

In component form (with $\vec{a} = a_x\hat{x} + a_y\hat{y} + a_z\hat{z}$)

$$\vec{a}\cdot\vec{b} = (a_x\hat{x} + a_y\hat{y} + a_z\hat{z})\cdot(b_x\hat{x} + b_y\hat{y} + b_z\hat{z}) =$$
$$= a_x b_x + a_y b_y + a_z b_z$$

(note: $\vec{a}^2 = a_x^2 + a_y^2 + a_z^2 = |\vec{a}|^2 = a^2$)

$$\vec{a}\times\vec{b} = (a_y b_z - a_z b_y)\hat{x} + (a_z b_x - a_x b_z)\hat{y} + (a_x b_y - a_y b_x)\hat{z}$$

or $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$

remember
$\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$

Special vectors:

- position $\vec{r} \equiv x\hat{x} + y\hat{y} + z\hat{z}$
- infinitesimal displacement $d\vec{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$
- radial unit vector $\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{\vec{r}}{r}$

$$r = \sqrt{x^2+y^2+z^2}$$

Differential calculus

Gradient of a scalar function $f = f(x,y,z) = f(\vec{r})$

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$$

gives change under infinitesimal displacement $d\vec{l}$

$$df = (\vec{\nabla} f)\cdot d\vec{l} = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

Ex. $\vec{\nabla} r = \vec{\nabla}\sqrt{x^2+y^2+z^2} = \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2+z^2}}\hat{x} + \frac{1}{2}\frac{2y}{\sqrt{x^2+y^2+z^2}}\hat{y}$

$$+ \frac{1}{2}\frac{2z}{\sqrt{x^2+y^2+z^2}}\hat{z} = \frac{x\hat{x}+y\hat{y}+z\hat{z}}{\sqrt{x^2+y^2+z^2}} = \frac{\vec{r}}{r} = \hat{r}$$

cf. $\vec{\nabla} x = \hat{x}$
(spher. coord.)

Think of $\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$ as an operator which can act on both scalar and vector functions

- gradient $\vec{\nabla} f = \hat{x}\frac{\partial}{\partial x}f + \hat{y}\frac{\partial}{\partial y}f + \hat{z}\frac{\partial}{\partial y}f$
- divergence $\vec{\nabla}\cdot\vec{a} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot(a_x\hat{x} + a_y\hat{y} + a_z\hat{z})$

$$= \frac{\partial}{\partial x}a_x + \frac{\partial}{\partial y}a_y + \frac{\partial}{\partial z}a_z$$

- curl

$$\vec{\nabla}\times\vec{a} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix} =$$

$$= \hat{x}\left(\frac{\partial}{\partial y}a_z - \frac{\partial}{\partial z}a_y\right) + \hat{y}\left(\frac{\partial}{\partial z}a_x - \frac{\partial}{\partial x}a_z\right) + \hat{z}\left(\frac{\partial}{\partial x}a_y - \frac{\partial}{\partial y}a_x\right)$$

(beware: $\vec{\nabla}$ behaves in many ways as a vector, but remember that it is a differential operator)

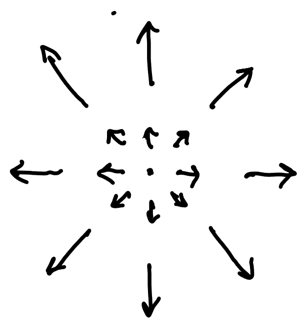
Geometrical interpretation

- gradient: points in direction of maximal change, magnitude gives rate of change
  $\vec{\nabla} f = 0$ at local max or min
- divergence: $\vec{\nabla}\cdot\vec{a} \neq 0$ where vector-fcn $\vec{a}$ has source or sink
- curl: measure of how much a vector fcn rotates

Ex. $\vec{a} = \vec{r}$

$\vec{\nabla}\cdot\vec{r} = 3$

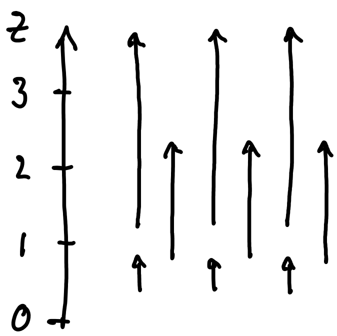
$\vec{\nabla}\times\vec{r} = \vec{0}$



$\vec{a} = z\hat{z}$

$\vec{\nabla}\cdot\vec{a} = 1$

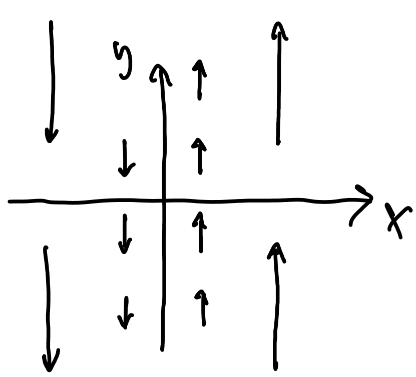
$\vec{\nabla}\times\vec{a} = \vec{0}$



$\vec{a} = x\hat{y}$

$\vec{\nabla}\cdot\vec{a} = 0$

$\vec{\nabla}\times\vec{a} = \hat{z}$



Product rules for scalar (f,g) and vector $(\vec{a},\vec{b})$ fcns:

$$\vec{\nabla}(fg) = f(\vec{\nabla}g) + g(\vec{\nabla}f)$$

$$\vec{\nabla}(\vec{a}\cdot\vec{b}) = \vec{a}\times(\vec{\nabla}\times\vec{b}) + \vec{b}\times(\vec{\nabla}\times\vec{a}) + (\vec{a}\cdot\vec{\nabla})\vec{b} + (\vec{b}\cdot\vec{\nabla})\vec{a}$$

$$\vec{\nabla}\cdot(f\vec{a}) = f(\vec{\nabla}\cdot\vec{a}) + (\vec{\nabla}f)\cdot\vec{a}$$

$$\vec{\nabla}\cdot(\vec{a}\times\vec{b}) = \vec{b}\cdot(\vec{\nabla}\times\vec{a}) - \vec{a}\cdot(\vec{\nabla}\times\vec{b})$$

$$\vec{\nabla}\times(f\vec{a}) = f(\vec{\nabla}\times\vec{a}) - \vec{a}\times(\vec{\nabla}f)$$
$$= f(\vec{\nabla}\times\vec{a}) + (\vec{\nabla}f)\times\vec{a}$$

$$\vec{\nabla}\times(\vec{a}\times\vec{b}) = (\vec{b}\cdot\vec{\nabla})\vec{a} - (\vec{a}\cdot\vec{\nabla})\vec{b} + \vec{a}(\vec{\nabla}\cdot\vec{b}) - \vec{b}(\vec{\nabla}\cdot\vec{a})$$

where $\vec{a}\cdot\vec{\nabla}$ is a scalar operator

$$(a_x\hat{x} + a_y\hat{y} + a_z\hat{z})\cdot\left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) =$$
$$= a_x\frac{\partial}{\partial x} + a_y\frac{\partial}{\partial y} + a_z\frac{\partial}{\partial z}$$

so that

$$(\vec{a}\cdot\vec{\nabla})\vec{b} = \left(a_x\frac{\partial}{\partial x}b_x + a_y\frac{\partial}{\partial y}b_x + a_z\frac{\partial}{\partial z}b_x\right)\hat{x} + \dots$$

Ex. (probl 1.22 b)

$$(\hat{r}\cdot\vec{\nabla})\hat{r} = \left(\left(\frac{x}{r}\hat{x} + \frac{y}{r}\hat{y} + \frac{z}{r}\hat{z}\right)\cdot\left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\right)\hat{r}$$

$$= \frac{1}{r}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)\frac{x\hat{x} + y\hat{y} + z\hat{z}}{\sqrt{x^2+y^2+z^2}} =$$

consider x-component

$$[(\hat{r}\cdot\vec{\nabla})\hat{r}]_x = \frac{1}{r}\left(x\left(\frac{\partial}{\partial x}x\right)\frac{1}{r} + x^2\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)\right. +$$

$$+ \; yx \frac{\partial}{\partial y}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) + zx \frac{\partial}{\partial z}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)\Bigg) =$$

$$= \frac{1}{r}\left(\frac{x}{r} + x^2\left(-\frac{1}{2}\frac{2x}{(x^2+y^2+z^2)^{3/2}}\right) + yx\left(-\frac{1}{2}\frac{2y}{(\quad)^{3/2}}\right) + \right.$$

$$\left. zx\left(-\frac{1}{2}\frac{2z}{(\quad)^{3/2}}\right)\right) =$$

$$= \frac{1}{r}\left(\frac{x}{r} - x\frac{x^2}{r^3} - x\frac{y^2}{r^3} - x\frac{z^2}{r^3}\right) = \frac{x}{r^2}\left(1 - \frac{x^2+y^2+z^2}{r^2}\right) = 0$$

the same holds for y- and z-components

$\therefore \; (\hat{r}\cdot\vec{\nabla})\,\hat{r} = \vec{0}$ (again much simpler in spher. coordinates)

Second derivatives

$$\vec{\nabla}\cdot(\vec{\nabla}f) = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}\right) =$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f \equiv \nabla^2 f$$

Laplacian

also $\nabla^2\vec{a} = (\nabla^2 a_x)\,\hat{x} + (\nabla^2 a_y)\,\hat{y} + (\nabla^2 a_z)\,\hat{z}$

$\vec{\nabla}\times(\vec{\nabla}f) = 0$ (follows from commuting

$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{a}) = 0$ derivatives: $\frac{\partial}{\partial x}\frac{\partial}{\partial y}a_x = \frac{\partial}{\partial y}\frac{\partial}{\partial x}a_x$)

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{a}) = \underbrace{\vec{\nabla}(\vec{\nabla}\cdot\vec{a})}_{\neq\,\nabla^2\vec{a}\;!} - \nabla^2\vec{a}$$

Note: ∇^2 sometimes written $\vec{\nabla}^2$ but strictly speaking this is an abuse of notation

Integral calculus

1-dim calc. In general the integral $\int_{x_1}^{x_2} g(x)\,dx$
depends on the value of g everywhere in $x \in [x_1, x_2]$
But if $g(x) = \frac{df}{dx}$ then the integral relates
derivative of fcn with its value on the boundaries.

$$\int_{x_1}^{x_2} \frac{df}{dx}\,dx = f(x_2) - f(x_1)$$, depends only on f at x_1 and x_2

generalises to several dimensions

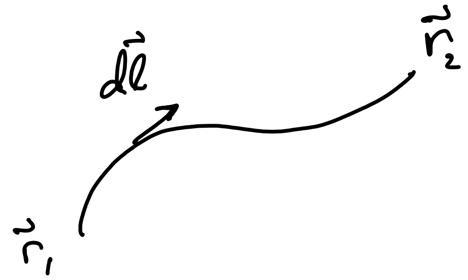
We are interested in
line, surface, and volume integrals:

$$\int_{\vec{r}_1}^{\vec{r}_2} \vec{a}\cdot d\vec{l} \;, \quad \int_S \vec{a}\cdot d\vec{S} \;, \quad \int_V f\,d\tau$$

closed integrals indicated by $\oint$

Line integrals:

$$\int_{\vec{r}_1}^{\vec{r}_2} \vec{a}\cdot d\vec{l}$$

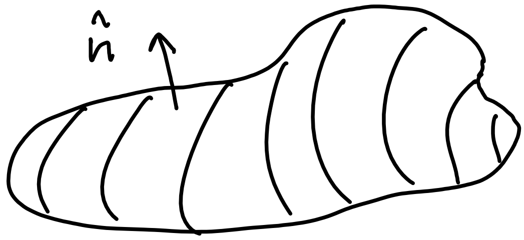


depends in general on $\vec{a}$ along the path (P) taken
but if $\vec{a} = \vec{\nabla} f$ then $(\vec{\nabla} f)\cdot d\vec{l} = df$ and

$$\int_{\vec{r}_1}^{\vec{r}_2} \vec{\nabla} f\cdot d\vec{l} = f(\vec{r}_2) - f(\vec{r}_1) \;, \quad \oint \vec{\nabla} f\cdot d\vec{l} = 0$$

(Differential forms: f 0-form, $\int_P df = \int_{\partial P} f$)

Surface integral



$$\int_S \vec{a}\cdot d\vec{S} \quad , \quad d\vec{S} = \hat{n}\, dS$$

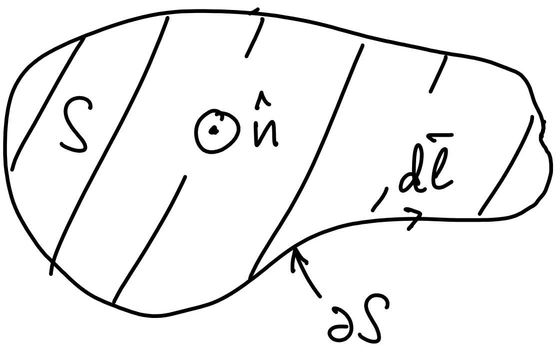
outward surface element

depends in general on the values of $\vec{a}$ on the surface, but if $\vec{a} = \vec{\nabla}\times\vec{b}$ then

$$\int_S (\vec{\nabla}\times\vec{b})\cdot d\vec{S} = \oint_{P=\partial S} \vec{b}\cdot d\vec{l}$$

Stokes' theorem

note: no dependence on shape of S only its boundary



relation of $\hat{n}$ and $d\bar{l}$ by right-hand rule

($\omega = \vec{b}\cdot d\vec{l}$ called 1-form , $\int_S d\omega = \int_{\partial S}\omega$)

Volume integral

$$\int_V f\, d\tau$$

volume element $d\tau = dx\,dy\,dz$

depends in general on values of f inside volume but if $f = \vec{\nabla}\cdot\vec{a}$ then

$$\int_V (\vec{\nabla}\cdot\vec{a})\, d\tau = \oint_{S=\partial V} \vec{a}\cdot d\vec{S}$$

Gauss' theorem

($\eta = \vec{a}\cdot d\vec{S}$ called 2-form , $\int_V d\eta = \int_{\partial V}\eta$)

For each of the differential operators $\frac{d}{dx}$, gradient, curl and div, there is a corresponding fundamental theorem of calculus relating the integral of a differential of some function over a certain domain to an integral of the function over the boundary of that domain

Also gives alternative way of defining the curl and div operators:

By the mean value theorem

$$\int_S (\vec{\nabla} \times \vec{a}) \cdot d\vec{S} = \vec{A}(S) \cdot (\vec{\nabla} \times \vec{a}) = \oint_{\partial S} \vec{a} \cdot d\vec{l}$$

$$\Rightarrow (\vec{\nabla} \times \vec{a}) \cdot \hat{n} = \lim_{S \to \vec{r}} \frac{1}{A(S)} \oint_{\partial S} \vec{a} \cdot d\vec{l}$$

"circulation of $\vec{a}$ per unit area on boundary of surface perpendicular to $\hat{n}$"

$$\int_V (\vec{\nabla} \cdot \vec{a}) \, d\tau = V(\Omega) (\vec{\nabla} \cdot \vec{a}) = \oint_{\partial V} \vec{a} \cdot d\vec{S}$$

$$\Rightarrow \vec{\nabla} \cdot \vec{a} = \lim_{\Omega \to \vec{r}} \frac{1}{V(\Omega)} \oint_{\partial V} \vec{a} \cdot d\vec{S}$$

"flux of $\vec{a}$ out of volume per unit volume"

These fundamental theorems are also useful when integrating by parts exploiting the product rules for scalar and vector fcns.

For example

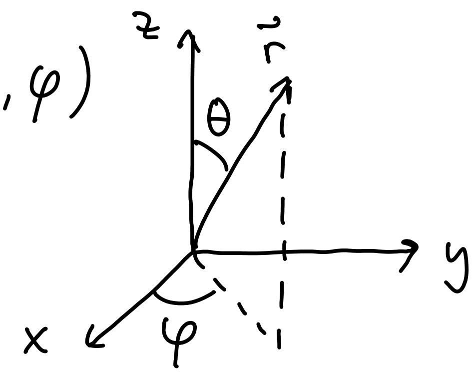
$$\int (\vec{\nabla} \cdot \vec{a}) f \, d\tau = \underbrace{\int \vec{\nabla} \cdot (f \vec{a}) \, d\tau}_{\oint_S f \vec{a} \cdot d\vec{S}} - \int \vec{a} \cdot \vec{\nabla} f \, d\tau$$

which we will use when calculating the energy of a continuous charge distribution

Curvilinear coordinate systems

"Apart from cartesian coordinates (x, y, z) we will also be using spherical (r, θ, φ) and cylindrical (s, φ, z) coordinates"

Spherical coordinates (r, θ, φ)



$$x = r \sin\theta \cos\varphi$$
$$y = r \sin\theta \sin\varphi$$
$$z = r \cos\theta$$

unit vectors – not constant!

$$\hat{r} = \sin\theta \cos\varphi \,\hat{x} + \sin\theta \sin\varphi \,\hat{y} + \cos\theta \,\hat{z}$$
$$\hat{\theta} = \cos\theta \cos\varphi \,\hat{x} + \cos\theta \sin\varphi \,\hat{y} - \sin\theta \,\hat{z}$$
$$\hat{\varphi} = -\sin\varphi \,\hat{x} + \cos\varphi \,\hat{y}$$

$(\hat{r}, \hat{\theta}, \hat{\varphi})$ forms right-handed system: $\hat{r} \times \hat{\theta} = \hat{\varphi}$ etc

Cylindrical coordinates (s, φ, z)

$$x = s \cos\varphi$$
$$y = s \sin\varphi$$
$$z = z$$

unit vectors

$$\hat{s} = \cos\varphi \,\hat{x} + \sin\varphi \,\hat{y}$$
$$\hat{\varphi} = -\sin\varphi \,\hat{x} + \cos\varphi \,\hat{y}$$
$$\hat{z} = \hat{z}$$

$(\hat{s}, \hat{\varphi}, \hat{z})$ right-handed system

Need to know line ($d\vec{l}$), surface ($d\vec{S}$) and volume ($d\tau$) elements as well as differential operators $\vec{\nabla} f$, $\vec{\nabla}\cdot\vec{a}$, $\vec{\nabla}\times\vec{a}$ and $\vec{\nabla}^2 f$

Consider general coordinate syst. $[u, v, w]$ with local one-to-one correspondence to (x, y, z)

$$x = x(u, v, w) \quad , \; y = y(\quad) \quad , \; z = z(\quad)$$

and $(\hat{u}, \hat{v}, \hat{w})$ right-handed orthogonal system

position vector

$$\vec{r} = x(u,v,w)\,\hat{i} + y(u,v,w)\,\hat{j} + z(u,v,w)\,\hat{k}$$

define scale factors

$$\frac{\partial \vec{r}}{\partial u} = \frac{\partial x}{\partial u}\hat{i} + \frac{\partial y}{\partial u}\hat{j} + \frac{\partial z}{\partial u}\hat{k} \equiv \left|\frac{\partial \vec{r}}{\partial u}\right| \hat{u} \equiv h_u\,\hat{u}$$

$$\frac{\partial \vec{r}}{\partial v} = \dots \quad , \quad \frac{d\vec{r}}{\partial w} = \dots$$

then $\frac{\partial \vec{r}}{\partial u} du = h_u\, du\, \hat{u}$ etc and

$$d\vec{r} = \frac{\partial \vec{r}}{\partial u} du + \frac{\partial \vec{r}}{\partial v} dv + \frac{\partial \vec{r}}{\partial w} dw = h_u du\,\hat{u} + h_v dv\,\hat{v} + h_w dw\,\hat{w}$$

$$= dl_u\,\hat{u} + dl_v\,\hat{v} + dl_w\,\hat{w}$$

the line, surface and volume elements are then

$$dl_u = h_u\, du \quad , \text{etc}$$

$$dS_u = dl_v\, dl_w = h_v h_w\, dv\, dw \quad , \text{etc}$$

$$d\tau = dl_u\, dl_v\, dl_w = h_u h_v h_w\, du\, dv\, dw$$

Ex. spherical coord

$$\tilde{r} = r\sin\theta\cos\varphi\,\hat{i} + r\sin\theta\sin\varphi\,\hat{j} + r\cos\theta\,\hat{k}$$

$$\frac{\partial \tilde{r}}{\partial r} = \dots = \hat{r} \Rightarrow h_r = 1$$

$$\frac{\partial \tilde{r}}{\partial \theta} = \dots = r\,\hat{\theta} \Rightarrow h_\theta = r$$

$$\frac{\partial \tilde{r}}{\partial \varphi} = \dots = r\sin\theta\,\hat{\varphi} \Rightarrow h_\varphi = r\sin\theta$$

Cylindrical coord

$$\tilde{r} = s\cos\varphi\,\hat{i} + s\sin\varphi\,\hat{j} + z\,\hat{k}$$

$$\frac{\partial \tilde{r}}{\partial s} = \dots = \hat{s} \Rightarrow h_s = 1$$

$$\frac{\partial \tilde{r}}{\partial \varphi} = \dots = s\,\hat{\varphi} \Rightarrow h_\varphi = s$$

$$\frac{\partial \tilde{r}}{\partial z} = \hat{z} \Rightarrow h_z = 1$$

The gradient

expand as

$$\vec{\nabla} f = f_u \hat{u} + f_v \hat{v} + f_w \hat{w}$$

know that $df = \vec{\nabla} f \cdot \vec{dl}$ is invariant

$$df = \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv + \frac{\partial f}{\partial w} dw$$

$$= f_u h_u du + f_v h_v dv + f_w h_w dw$$

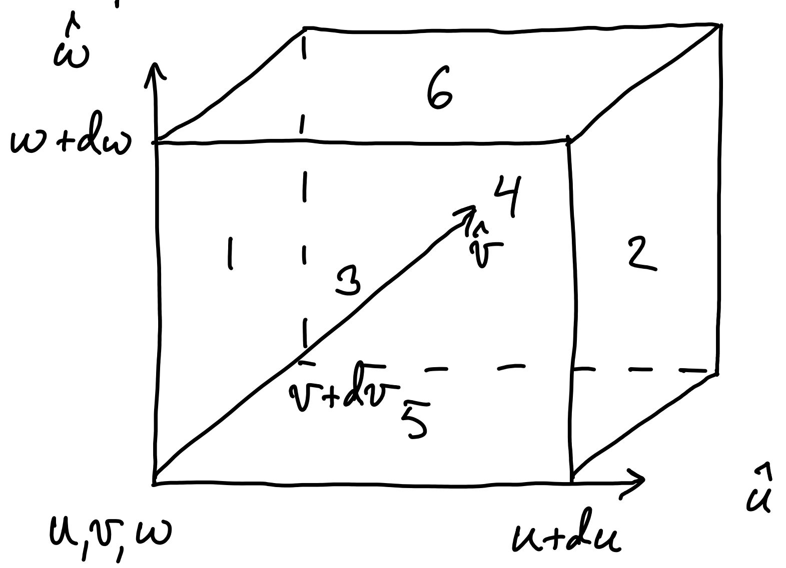
$$\Rightarrow f_u = \frac{1}{h_u} \frac{\partial f}{\partial u}, \quad f_v = \frac{1}{h_v} \frac{\partial f}{\partial v}, \quad f_w = \frac{1}{h_w} \frac{\partial f}{\partial w}$$

Divergence

Expand vector field as

$$\vec{a}(u,v,w) = a_u(u,v,w)\,\hat{u} + \ldots$$

Calc. flux out of volume $h_u du \, h_v dv \, h_w dw$



$$1+2: \quad a_u(u+du,v,w)\, h_v(u+du,v,w)\, h_w(u+du,v,w)\, dv\, dw$$

$$- a_u(u,v,w)\, h_v(u,v,w)\, h_w(u,v,w)\, dv\, dw =$$

$$= \frac{\partial}{\partial u}\left(a_u h_v h_w\right) du\, dv\, dw$$

$$3+4 = \frac{\partial}{\partial v}\left(a_v h_u h_w\right) du\, dv\, dw$$

$$5+6 = \frac{\partial}{\partial w}\left(a_w h_u h_v\right) du\, dv\, dw$$

gives flux per unit volume

$$\vec{\nabla}\cdot\vec{a} = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}\left(h_v h_w a_u\right) + \frac{\partial}{\partial v}\left(h_u h_w a_v\right) + \frac{\partial}{\partial w}\left(h_u h_v a_w\right)\right]$$

Curl

$f = u \Rightarrow \vec{\nabla} u = \frac{1}{h_u}\hat{u} \Rightarrow \hat{u} = h_u \vec{\nabla} u$

same way $\hat{v} = h_v \vec{\nabla} v$, $\hat{w} = h_w \vec{\nabla} w$

$\Rightarrow \vec{a} = a_u\hat{u} + a_v\hat{v} + a_w\hat{w} = a_u h_u \vec{\nabla} u + a_v h_v \vec{\nabla} v + \ldots$

use product rule $\vec{\nabla}\times(f\vec{b}) = f(\vec{\nabla}\times\vec{b}) + \vec{\nabla} f\times\vec{b}$

with $\vec{b} = \vec{\nabla} g \Rightarrow \vec{\nabla}\times(f\vec{\nabla} g) = f(\underbrace{\vec{\nabla}\times\vec{\nabla} g}_{=0}) + \vec{\nabla} f\times\vec{\nabla} g$

$$\Rightarrow \vec{\nabla}\times(a_u\hat{u}) = \vec{\nabla}\times(a_u h_u \vec{\nabla} u) = \vec{\nabla}(a_u h_u)\times\vec{\nabla} u$$

$$= \left[\frac{1}{h_u}\frac{\partial}{\partial u}(a_u h_u)\hat{u} + \frac{1}{h_v}\frac{\partial}{\partial v}(a_u h_u)\hat{v} + \ldots\right]\times\frac{1}{h_u}\hat{u}$$

$$= 0 + \frac{1}{h_u h_v}\frac{\partial}{\partial v}(a_u h_u)\underbrace{\hat{v}\times\hat{u}}_{-\hat{w}} + \frac{1}{h_u h_w}\frac{\partial}{\partial w}(a_u h_u)\underbrace{\hat{w}\times\hat{u}}_{\hat{v}}$$

similarly for $\vec{\nabla}\times(a_v\hat{v})$, $\vec{\nabla}\times(a_w\hat{w})$

$$\Rightarrow \vec{\nabla}\times\vec{a} = \frac{1}{h_u h_v h_w}\begin{vmatrix} h_u\hat{u} & h_v\hat{v} & h_w\hat{w} \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ a_u h_u & a_v h_v & a_w h_w \end{vmatrix}$$

Laplace

$$\nabla^2 f = \vec{\nabla} \cdot \vec{\nabla} f = \vec{\nabla} \cdot \left(\frac{1}{h_u} \frac{\partial f}{\partial u} \hat{u} + \frac{1}{h_v} \frac{\partial f}{\partial v} \hat{v} + \frac{1}{h_w} \frac{\partial f}{\partial w} \hat{w} \right) =$$

$$= \frac{1}{h_u h_v h_w} \left[\frac{\partial}{\partial u} \left(h_v h_w \frac{1}{h_u} \frac{\partial f}{\partial u} \right) + \frac{\partial}{\partial v} \left(h_u h_w \frac{1}{h_v} \frac{\partial f}{\partial v} \right) + \frac{\partial}{\partial w} \left(h_u h_v \frac{1}{h_w} \frac{\partial f}{\partial w} \right) \right]$$

Dirac delta function

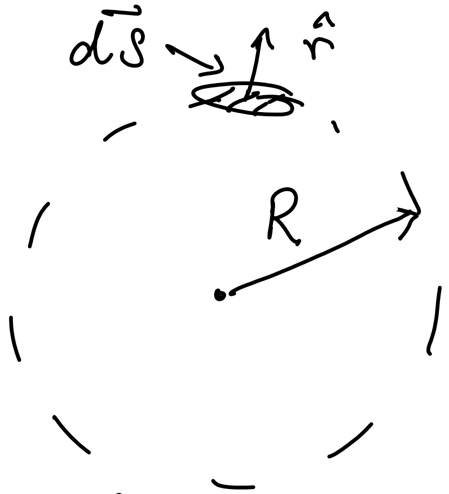
Try to calculate divergence of $\vec{a} = \frac{\hat{r}}{r^2}$

for $r \neq 0$ we can use

$$\vec{\nabla} \cdot (a_r \hat{r}) = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 a_r) \Rightarrow \vec{\nabla} \cdot \left(\frac{\hat{r}}{r^2} \right) = 0$$

but using Gauss' theorem to integrate $\vec{a}$ over a sphere with radius R we get

$$\int_V \vec{\nabla} \cdot \left(\frac{\hat{r}}{r^2} \right) d\tau = \oint_{\partial V} \frac{\hat{r}}{R^2} \cdot \underset{\uparrow \atop \hat{r} R^2 \sin\theta \, d\theta \, d\varphi}{d\vec{S}} =$$



$$= \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\varphi = 4\pi$$

so there must be a contribution from the origin $\vec{r} = (0,0,0)$ to $\vec{\nabla} \cdot \vec{a}$ to make up the difference - called Dirac delta function

In one dimension it is defined by

$$\int_{-\infty}^{\infty} f(x) \, \delta(x-a) \, dx = f(a)$$

can be thought of as limit of Gaussian

$$\delta(x-a) = \lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{(x-a)^2}{2\sigma^2} \right]$$

important property

$$\int f(x)\,\delta[g(x)]\,dx = \int f(x)\,\frac{1}{|g'(x)|_{g(x=a)=0}}\,\delta(x-a)\,dx$$

and in 3 dimensions

$$\delta^{(3)}(\vec{r}) = \delta(x)\,\delta(y)\,\delta(z)$$

It then follows that

$$\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\,\delta^3(\vec{r})$$

corresponding to a point source at the origin

<u>Helmholtz theorem</u>

From Maxwell's eqn's we get $\vec{\nabla}\cdot\vec{E}$ and $\vec{\nabla}\times\vec{E}$, does this specify $\vec{E}$ completely?

For this to be true the following conditions must be fulfilled

$\vec{\nabla}\cdot\vec{E} \to 0$ faster than $\frac{1}{r^2}$ as $r\to\infty$

$\vec{\nabla}\times\vec{E} \to 0$ faster than $\frac{1}{r^2}$ as $r\to\infty$

$\vec{E} \to 0$ as $r\to\infty$